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Methodus Differentialis:

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TRACTATUS

DE

SUMMATIONE

ET

INTERPOLATIONE

SERIERUM INFINITARUM.

AUCTORE *JACOBO STIRLING*, R. S. S.

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PRÆFATIO.

CUM ſæpiſſime eveniat ut Series adeo lente convergant ad veritatem, ut non magis conducant ad finem propoſitum quam ſi revera eſſent divergentes; exhibui quædam Theoremata in parte priore hujus Tractatus, quibus prompte acceditur ad valores earum Serierum quæ omnium lentiſſime approximant: Et eo quidem conſilio, ut Problemata quæ dependent ex Quadraturis, eodem jure pro ſolutis haberi poſſint quo ea quæ reducuntur ad Æquationes affectas. Hic enim non inquiro quænam Series ſint ſummabiles (ut ex titulo nonnulli forſan conjiciant) ſed quibus artibus aſſequamur valores earum quæ ſummari nequeunt.

Hujus generis Tractatum edidit Jacobus Gregorius *in Appendice ad* Veram Circuli & Hyperbolæ Quadraturam, *ubi docuit modum facillimum approximandi ad Areas harum Curvarum ex datis perpaucis polygonis pauciorum laterum. Eoque pacto methodum Exhauſtionum*

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Archimedis *expeditam adeo reddidit, ut ſi eadem methodus pari cum ſucceſſu ad reliquas Curvas extendi poſſet, fruſtra impenderetur ulterior opera in eliciendis Areis. Quodque* Gregorius *ibi effecit in Serie polygonorum, idem fere nos hic generaliter proponimus in qualibet aliâ quæ ſimplici gaudet Terminorum relatione.*

Newtonum *hâc de re olim meditatum fuiſſe conſtat ex Epiſtolâ ejus priori ad* Oldenburgum *quæ impreſſa habetur in* Commercio Epiſtolico Collinii. *Nimirum poſtquam computo produxerat Aream Circuli ad ſedecim decimales, inquit,* Si alias artes adhibuiſſem, potui per eundem numerum Terminorum Seriei, perveniſſe ad multo plura loca figurarum, puta viginti quinque aut amplius: Sed animus fuit hic oſtendere quid per ſimplex Seriei computum præſtari poſſet. *Attamen inter ſcripta ejus (hactenus ſaltem edita) ne veſtigium quidem extat unde conjecturam faciamus de hiſce artibus, etiamſi inſignem eas proferendi occaſionem nactus eſſet in hâc ipſiſſimâ Epiſtolâ. In priori etiam Epiſtolâ ad eundem, ibidem impreſſâ; narrat ſe excogitaſſe quædam circa Reductionem infinitarum Serierum in finitas ubi rei natura tulerit: Quæ ſi inter opera ejus poſthuma ſuperſint, proculdubio huic doctrinæ lucem haud exiguam præbebunt: Nam Theoremata generalia quæ valores Serierum exhibent accurate, quando id fieri poteſt, in aliis caſibus neceſſe approximabunt, modo rite adhibeantur.*

Principium quod in hunc effectum vulgo adducitur, eſt Aſſumptio differentiæ duorum valorum ſucceſſivorum alicujus quantitatis, ut exinde formentur Termini quorum ſumma prius nota fuerat; idem ſcilicet principium, quo olim uſus eſt Newtonus *ad Ordinatam Curvæ deducendam*

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ex data Area. Quod licet ſit univerſale in Quadraturis, eſt particulare tantummodo in Summationibus; quippe ad eas ſolummodo Series applicabile quarum Termini poſſunt aſſignari: quum tamen æque facilis ſit aſſignatio Summæ atque Termini in iis Seriebus quæ magnâ ex parte in Quadraturis prodire ſolent.

Fundamentum longe generalius ſubminiſtravit Methodus Newtoni Differentialis: *Ille utique deſcribit Curvam Parabolicam per extremitates quotlibet Ordinatarum ſive Terminorum; eâque ratione aſſignat valorem cujuſvis intermedii per Seriem infinitam; quæ tamen ad veritatem non accedet ſi Terminus ille longe diſtet ab initio. Ut obtinerem igitur Terminos Serierum remotiſſimos, deſcripſi figuram Hyperbolicam per extremitates Terminorum; & res ſucceſſit, prodeunte valore Termini quantumvis diſtantis, per Seriem convergentem. Sed & hoc Problemate generaliter ſoluto, non latuit caſus ejus facillimus, utpote inventio Termini infinito intervallo diſtantis a principio; quæ quidem æquipollet Summationi Serierum. Cæterum deſcriptio Curvæ cujuſcunque Geometricæ per data puncta, uni tantum Serierum generi ſufficit: atque alia ſunt innumera nullatenus tractanda ex hoc fundamento. Nam valor Termini per Parabolam aut Hyperbolam inventus, non approximat niſi ubi differentiæ ſecundum præſcripta* Newtoniana *ſumptæ conſtituant progreſſionem celeriter ſatis decreſcentem.*

Hiſce perſpectis, me tandem contuli ad contemplandam Relationem Terminorum, proprietatem Serierum maxime inſignem & ſimplicem, quæ vulgo ad eas continuandas adhiberi ſolet: Non enim ignorabam Moivræum *hanc Terminorum proprietatem ſummo cum ſucceſſu in Algebram*

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bram introduxiſſe, tanquam fundamentum ad difficillima circa Series recurrentes Problemata ſolvenda: Ideoque decreveram experiri num ad alias quoque extendi poſſet: Quod ſane dubitabam cum tanta ſit inter recurrentes atque alias Series diſcrepantia. At facto experimento, res præter ſpem ſucceſſit, deprehendi enim Inventum hocce Moivræanum *continere principia generaliſſima & etiam ſimpliciſſima non ſolum in Seriebus recurrentibus, ſed & in quibuſlibet aliis, in quibus relatio Terminorum variatur ſecundum legem quamcunque regularem. Nam Terminorum relatio, etiamſi ſit variabilis, facile tamen aſſignabilis eſt: Atque exinde Summationes & Interpolationes, aliaque ejuſmodi Problemata difficiliora perducuntur ad quoddam genus Æquationum, quæ præter Radicem extrahendam involvunt alias quantitates incognitas quæ nequeunt eliminari; quo non obſtante, harum Æquationum reſolutio perficitur nonnunquam ſummâ facilitate, nonnunquam vero non ſuccedit niſi mediantibus* Moivræi *Inventis circa aſſignationem Terminorum in Seriebus recurrentibus. Et de hoc argumento totus fere agit hicce Libellus.*

Problema de invenienda Unciâ mediâ in permagnâ dignitate binomii ſolutum erat a Moivræo *ante aliquot annos quam ego idem attigeram: Nec probabile eſt quod in hunc uſque diem de eodem cogitaſſem, ni ſuggeſſiſſet Spectatiſſimus Vir, Dr. *Alex. Cuming, *ſe plurimum ſuſpicari an idem ſolvi poſſet per* Methodum Differentialem Newtoni.

TRACTATUS

TRACTATUS
DE
SUMMATIONE & INTERPOLATIONE
Serierum Infinitarum.

INTRODUCTIO.

SICUT Curvæ non determinantur ex datis quotlibet earum ordinatis, ſed ex relatione inter Abſciſſas & Ordinatas in genere; ita neque Series determinantur ex datis quotlibet earum Terminis, ſed ex relatione inter Terminos ſucceſſivos. Quantitates enim quotcunque, numero tamen finitæ, conſtituere poſſunt Terminos in diverſis Seriebus: unica vero eſt Series in qua iidem ſunt Termini initiales, eademque lex formandi reliquos in infinitum. Relationes igitur Terminorum primo loco indagandæ ſunt; deinde hæ inventæ denotandæ ſunt Æquationibus Differentialibus perinde ac *Carteſius* definivit Curvas Æquationibus Algebraicis: quibus habitis Problemata

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circa Summationem & Interpolationem, aliaque iſtiuſmodi ad Series ſpectantia ſolventur per Analyſin non minus certam quam eſt Algebra vulgaris.

De Relatione Terminorum.

Termini Serierum binatim, ternatim, aut plures numero ſumpti relationem quandam magna ex parte inter ſe obtinent ſimplicem & obviam, per quam Series determinantur & producuntur ad libitum. Ut ſi dividatur unitas per $1-x$, prodibit progreſſio Geometrica in quâ Terminus quivis conſequens eſt ad illum immediate antecedentem ut x ad unitatem. Per hanc enim Proprietatem Series $1+x+x^2+x^3+x^4+x^5+$ &c. ab alia quavis diſtinguitur & produciur in infinitum.

Proponatur fractio $\frac{1}{r+sx+txx}$ in Seriem reſolvenda: illum in finem finge $y=\frac{1}{r+sx+txx}$; ac ducendo utramque partem in Denominatorem, habebitur $y\times\overline{r+sx+txx}=1$, vel $y\times\overline{r+sx+txx}-1=0$. Pro y ſubſtitue Seriem debitæ formæ $A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+$ &c. atque reſultabit

$$\left.\begin{matrix} rA+rB \\ -1+sA \end{matrix}\right\} x \left.\begin{matrix} +rC \\ +sB \\ +tA \end{matrix}\right\} x^2 \left.\begin{matrix} +rD \\ +sC \\ +tB \end{matrix}\right\} x^3 \left.\begin{matrix} +rE \\ +sD \\ +tC \end{matrix}\right\} x^4 \left.\begin{matrix} +rF \\ +sE \\ +tD \end{matrix}\right\} x^5 + \&c. = 0.$$

Ubi ponendo membra homologa nihilo æqualia, ad determinandum Coefficientes aſſumptos, erit $rA-1=0$, $rB+sA=0$; deinde $rC+sB+tA=0$, $rD+sC+tB=0$, $rE+sD+tC=0$, & ſic in infinitum. Ex quibus conſtat eandem ubique prodire relationem inter tres quoſcunque Terminos ſucceſſivos. Similiter uſurpando pro y Seriem hujus formæ $Ax^{-2}+Bx^{-3}+Cx^{-4}+Dx^{-5}+$ &c. & ſubſtituendo eandem in Æquatione, reſultabit

$$\left.\begin{matrix} tA+tB \\ -1+sA \end{matrix}\right\} x^{-1} \left.\begin{matrix} +tC \\ +sB \\ +rA \end{matrix}\right\} x^{-2} \left.\begin{matrix} +tD \\ +sC \\ +rB \end{matrix}\right\} x^{-3} \left.\begin{matrix} +tE \\ +sD \\ +rC \end{matrix}\right\} x^{-4} + \&c. = 0.$$

Atque jam ad determinandos Coefficientes, habemus Æquationes $tA-1=0$, $tB+sA=0$; tum $rA+sB+tC=0$, $rB+sC+tD=0$, $rC+sD+tE=0$, & ita deinceps in reliquis. Ubi patet eaſdem prodire relationes ac in computo ſuperiore, ſumptis tamen Coefficientibus in ordine inverſo. Et quantitates r, s, t, quæ indicant relationem Terminorum, eædem ſunt quæ in Denominatore fractionis.
Hanc

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Hanc autem harum Serierum Proprietatem, utcunque obvia ſit, primus omnium in uſus adhibuit D. *Moivræus* in ſolutione Problematum circa Series infinitas, quæ aliàs forent intricatiſſima.

In plurimis autem Seriebus relatio Terminorum non eſt conſtans ut in iis ex Diviſione prodeuntibus; variatur tamen ſæpiſſime ſecundum notam legem oculi intuitu patentem; cujus rei Exempla ſunt Series quæ pro Quadraturis vulgo proferuntur, aliaque innumera. Ut in hac $1 + \frac{2}{3}x + \frac{8}{15}x^2 + \frac{16}{35}x^3 + \frac{128}{315}x^4 +$ &c. Termini continuantur in infinitum per multiplicationem continuam harum fractionum $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$, $\frac{8}{9}$, &c. Et in hac $1 + \frac{1}{6}x + \frac{3}{40}x^2 + \frac{5}{112}x^3 + \frac{35}{1152}x^4 +$ &c. per multiplicationem harum $\frac{1\times1}{2\times3}$, $\frac{3\times3}{4\times5}$, $\frac{5\times5}{6\times7}$, $\frac{7\times7}{8\times9}$, &c. Et hæ fractiones variantur ſecundum legem cuivis manifeſtam; atque adeo in ipſarum aſſignatione, nulla erit difficultas.

De Æquationibus Differentialibus quæ definiunt Series.

Æquatio Seriem definiens ea eſt, quæ aſſignat relationem Terminorum generaliter ex datis ipſorum diſtantiis ab initio. Concipiendi autem ſunt Termini conſiſtere ſuper rectam poſitione datam tanquam totidem Ordinatæ, quarum diſtantia communis eſt unitas. Unitatem vero ſimplicitatis gratiâ ubique uſurpo in ſequentibus pro intervallo communi: quod ſemel monuiſſe ſufficiat.

Terminos Seriei initiales deſigno literis Alphabeti initialibus A, B, C, D, *&c.* A eſt primus, B ſecundus, C tertius, & ſic porro. Et Terminum quemvis in genere denoto literâ T, atque reliquos ordine ſuccedentes eâdem literâ, adjunctis numeris *Romanis* I, II, III, IV, V, VI, VII, &c. diſtinctionis gratiâ. Ut ſi T ſit decimus, erit T′ decimus primus, T″ decimus ſecundus, T‴ decimus tertius, & ſic deinceps. Et in genere quicunque Terminus definitur per T, ſuccedentes definientur univerſaliter per T′, T″, T‴, T^{iv}, &c.

Diſtantiam Termini T à Termino quovis dato, vel à puncto quovis dato inter duos quoſlibet Terminos intermedio, denoto quantitate indeterminatâ z: quo pacto diſtantiæ Terminorum T′, T″, T‴, &c. à prædicto Termino vel puncto, erunt $z+1$, $z+2$, $z+3$, &c. Nam incrementum Abſciſſæ z æquale eſt intervallo communi Terminorum ſupra Abſciſſam conſiſtentium: atque quantitates z, $z+1$, $z+2$, $z+3$, &c. ſibi mutuò ſuccedunt dum Termini poſteriores ſuccedunt anterioribus.

Hiſce præmiſſis, proponatur Series 1, $\frac{1}{2}x$, $\frac{1}{8}x^2$, $\frac{1}{48}x^3$, $\frac{1}{384}x^4$, $\frac{1}{3840}x^5$, &c. ubi relationes Terminorum ſunt $B = \frac{1}{2}Ax$, $C = \frac{1}{4}Bx$, $D = \frac{1}{6}Cx$,

E =

$E = \frac{7}{8} Dx$, &c. relatio in genere definietur Æquatione $T' = \frac{z + \frac{1}{2}}{z + 1} Tx$, ubi z deſignat diſtantiam ipſius T à primo Termino Seriei. Nam ſcribendo 0, 1, 2, 3, 4, &c. ſucceſſive pro z, prodibunt relationes Terminorum in Serie propoſita. Similiter ſi z denotet diſtantiam ipſius T à ſecundo Termino Seriei, Æquatio erit $T' = \frac{z + \frac{3}{2}}{z + 2} Tx$, uti conſtabit ſcribendo numeros — 1, 0, 1, 2, 3, &c. ſucceſſive pro z. Vel ſi indeterminata z denotet locum Termini T in Serie, ejus valores ſucceſſivi erunt 1, 2, 3, 4, &c. & Æquatio $T' = \frac{z - \frac{1}{2}}{z} Tx$; ut experienti patebit.

Innumeræ igitur diverſæ Æquationes Differentiales eandem Seriem definire poſſunt, prout in hoc vel illo puncto ſumitur initium Abſciſſæ z. Et contra eadem Æquatio definit innumeras Series diverſas, adhibendo diverſos valores ſucceſſivos pro z. Nam in Æquatione $T' = \frac{z - \frac{1}{2}}{z} Tx$, quæ definit Seriem de qua nunc egimus, quando 1, 2, 3, 4, &c. ſunt valores Abſciſſæ ordine ſuccedentes; ſcribe $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, &c. ſucceſſive pro z, & provenient relationes Terminorum $B = \frac{2}{3} Ax$, $C = \frac{4}{5} Bx$, $D = \frac{6}{7} Cx$, &c. Unde fit Series A, $\frac{2}{3} Ax$, $\frac{8}{15} Ax^2$, $\frac{16}{35} Ax^3$, $\frac{128}{315} Ax^4$, &c. quæ eſt diverſa a priore. Æquatio autem ſemper determinat Seriem ex datis valoribus Abſciſſæ & ſimul primo Termino, ubi Æquatio involvit duos tantum Terminos Seriei. Ut in noviſſima dantur omnes Termini ex dato primo A. Attamen ubi Æquatio involvit tres Terminos, oportet duos dari, & tres ubi involvit quatuor, & ſic porro ad determinandam Seriem.

Proponatur jam Series x, $\frac{1}{6} x^3$, $\frac{3}{40} x^5$, $\frac{5}{112} x^7$, $\frac{35}{1152} x^9$, &c. ubi relationes Terminorum ſunt $B = \frac{1 \times 1}{2 \times 3} Ax^2$, $C = \frac{3 \times 3}{4 \times 5} Bx^2$, $D = \frac{5 \times 5}{6 \times 7} Cx^2$, &c. Æquatio ad eandem erit $T' = \frac{\overline{2z - 1} \times \overline{2z - 1}}{2z \times \overline{2z + 1}} Tx^2$, vel $T' = \frac{4zz - 4z + 1}{4zz + 2z} Tx^2$, ubi valores ſucceſſivi indeterminatæ z ſunt 1, 2, 3, 4, &c. Igitur in Æquatione definiente Seriem, Abſciſſa z poteſt eſſe unius, duarum, vel plurium dimenſionum.

Series quarum Termini ſunt aſſignabiles, definiri poſſunt Æquationibus Terminos aſſignantibus. Sic Series $1 - \frac{1}{2} x + \frac{1}{3} x^2 - \frac{1}{4} x^3 + \frac{1}{5} x^4 -$ &c. definitur Æquatione $T = \frac{-x^z}{z + 1}$, ut conſtabit ſubſtituendo 0, 1,

2, 3,

2, 3, &c. pro z. Atque ad eundem modum Series $x+\frac{1}{4}x^2+\frac{1}{9}x^3+\frac{1}{16}x^4+\frac{1}{25}x^5+$ &c. denotatur per hanc $T=\frac{x^z}{zz}$. Tales Æquationes reduci ſemper poſſunt ad eas alterius generis: nam ubi Termini ſunt aſſignabiles, relationes eorum erunt etiam aſſignabiles. Et differentia inter has & illas raro tanta erit quin cuique tuto procedere licuerit prout ſibi viſum fuerit.

Ex hactenus dictis conſtat relationes Terminorum conſequentium derivari ex iis antecedentium, ſcribendo pro z ejus valorem ſucceſſivum $z+1$ in Æquatione Differentiali. Proponatur Æquatio $T'=\frac{z+n}{z}T$; ſcribe $z+1$ pro z, T' pro T, & T'' pro T'; atque orietur $T''=\frac{z+n+1}{z+1}T'$, quæ eſt relatio inter Terminos T' & T''. In hac ultima ſcribe valores variabilium conſequentes $z+1$, T'', & T''', pro antecedentibus z, T' & T'', & obtinebis $T'''=\frac{z+n+2}{z+2}T''$ relationem utique inter T'' & T'''.

Sed & contraria operatione regredi licet ad relationem Terminorum antecedentium ex data ea conſequentium. Sit Æquatio $T''=\frac{z^2-1}{z^3+3z^2+3z+2}T'$, & in eadem ſcribe T pro T', T' pro T'', & $z-1$ pro z; atque habebis $T'=\frac{zz-2z}{z^3+1}T$. Hoc modo regrediendo & progrediendo, poſſunt Series continuari hinc inde in infinitum ubi earum natura tulerit: atque etiamſi ignoretur quinam Termini denotentur per T, T', T'', &c. inſtituere licet ſuper iis computum, tanquam eſſent prorſus cogniti.

Æquationes de quibus hactenus egimus, involvunt duos tantum Terminos Seriei; poſſunt vero involvere plures, & tam Termini quam indeterminata z eſſe plurium dimenſionum. Ceterùm in hoc ſpecimine ſimpliciores tantum attingo.

De Forma & Reductione Serierum.

Poſtquam perduximus Series ad Æquationes Differentiales, monſtrandum eſt qua ratione eædem ſint reſolvendæ in numeris. Nam Analyſtæ munus eſt quantitates quocunque modo determinatas eruere accurate vel quam proxime. Radices autem Æquationum Differentialium commodiſſime reſolvuntur in Series formarum ſequentium

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$$A + Bz + Cz.\overline{z-1} + Dz.\overline{z-1}.\overline{z-2} + Ez.\overline{z-1}.\overline{z-2}.\overline{z-3} + \&c.$$

$$A + \frac{B}{z} + \frac{C}{z.\overline{z+1}} + \frac{D}{z.\overline{z+1}.\overline{z+2}} + \frac{E}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Quippe ubi z eſt quantitas parva, prior forma erit adhibenda; & poſterior ubi magna. Et hæ Series quæ componuntur ex Factoribus in progreſſione Arithmetica, longe magis idoneæ ſunt huic negotio quam vulgares quæ conflantur ex dignitatibus indeterminatæ aſcendentibus vel deſcendentibus. Inſuper forma poſterior hoc habet commodi, quod in eadem poteſt eſſe z quantumvis fere magna, id quod efficit Seriem celerrime convergentem.

Sin vero harum Serierum formæ per quaſlibet operationes mutentur, debent revocari ad eaſdem quarum priùs erant, ut Termini reddantur homologi eorumque collatio inſtitui poſſit prout res exigit. Ut ſi habeatur Æquatio ſequens;

$$T = A + Bz + Cz.\overline{z-1} + Dz.\overline{z-1}.\overline{z-2} + Ez.\overline{z-1}.\overline{z-2}.\overline{z-3} + \&c.$$

Eadem ducta in z priſtinam formam amittet, atque novam induet, proveniente ſcilicet

$$Tz = Az + Bz^2 + Cz^2.\overline{z-1} + Dz^2.\overline{z-1}.\overline{z-2} + Ez^2.\overline{z-1}.\overline{z-2}.\overline{z-3} + (\&c.$$

Unde conſtat Terminos ejus comparari non poſſe cum correſpondentibus in Serie priore. Itaque ut debita forma reſtituatur, ſic operor

$$Az \text{ - - - - - - - - } = Az,$$
$$Bz^2 \text{ - - - - - - - } = Bz + Bz\overline{z-1},$$
$$Cz^2.\overline{z-1} \text{ - - - - - } = * \; 2Cz\overline{z-1} + Cz\overline{z-1}.\overline{z-2},$$
$$Dz^2.\overline{z-1}.\overline{z-2} \text{ - - } = * \quad * \quad 3Dz\overline{z-1}.\overline{z-2} + Dz\overline{z-1}.\overline{z-2}.\overline{z-3}$$
$$Ez^2.\overline{z-1}.\overline{z-2}.\overline{z-3} = * \quad * \text{ - - - - } * \text{ - - - } 4Ez.\overline{z-1}.\overline{z-2}.\overline{z-3} + \&c.$$

Adeoque colligendo Terminos homologos in unum, Series ad priſtinam formam reducta erit

$$Tz = \left.\begin{matrix} +A \\ +B \end{matrix}\right\} z \left.\begin{matrix} +B \\ +2C \end{matrix}\right\} z.\overline{z-1} \left.\begin{matrix} +C \\ +3D \end{matrix}\right\} z.\overline{z-1}.\overline{z-2} \left.\begin{matrix} +D \\ +4E \end{matrix}\right\} z.\overline{z-1}.\overline{z-2}.\overline{z-3} + \&c.$$

Nimirum cum Terminorum ὁμολογία neutiquam pendeat ex Coefficientibus A, B, C, D, &c. ſed omnino ex indeterminata z, Terminus $\left.\begin{matrix} +A \\ +B \end{matrix}\right\} z$ primus in hac Serie comparari poteſt cum Bz ſecundo in altera, item ſecundus in hac cum tertio in illa, & ſic in reliquis.

Similiter ſi in Æquatione priore $T = A + Bz + Cz.\overline{z-1} + Dz.\overline{z-1}.\overline{z-2} + \&c.$ ſcribantur variabilium valores ſuccedentes pro præſentibus, hoc eſt T' pro T, & $z+1$ pro z, emerget

$T' =$

$$T' = A + B\overline{z+1} + C\overline{z+1}.z + D\overline{z+1}.z.\overline{z-1} + E\overline{z+1}.z.\overline{z-1}.\overline{z-2} + \&c.$$

Eſt vero

$$\begin{array}{ll} A \text{ - - - - - - - - - } & = A, \\ B\overline{z+1} \text{ - - - - - - - } & = B + Bz, \\ C\overline{z+1}.z \text{ - - - - - } & = * \;\; 2Cz + Cz.\overline{z-1}, \\ D\overline{z+1}.z.\overline{z-1} \text{ - - } & = * \text{ - - } * \text{ - - } 3Dz.\overline{z-1} + Dz.\overline{z-1}.\overline{z-2} \\ E\overline{z+1}.z.\overline{z-1}.\overline{z-2} & = * \text{ - - } * \text{ - - - } * \text{ - - - - - } 4Ez.\overline{z-1}.\overline{z-2} + \&c. \end{array}$$

Et inde

$$T' = \left.\begin{matrix} +A + B \\ +B + 2C \end{matrix}\right\} z \left.\begin{matrix} + C \\ +3D \end{matrix}\right\} z.\overline{z-1} \left.\begin{matrix} + D \\ +4E \end{matrix}\right\} z.\overline{z-1}.\overline{z-2} \left.\begin{matrix} + E \\ +5F \end{matrix}\right\} z.\overline{z-1}.\overline{z-2}.\overline{z-3} + \&c.$$

Quæ eſt forma deſiderata.

Hem vero harum Operationum fundamentum. Quantitas reducenda, per multiplicationem reducatur ad Poteſtates Indeterminatæ z: dein operare ad modum Exempli ſequentis. Sit $\overline{z-3} \times z \times \overline{z+1} \times \overline{z+4}$ quantitas reducenda; finge

$$\overline{z-3} \times z \times \overline{z+1} \times \overline{z+4} = az.\overline{z-1}.\overline{z-2}.\overline{z-3} + bz.\overline{z-1}.\overline{z-2} + cz.\overline{z-1} + dz.$$

Ubi maximus numerus Factorum in quantitate reſoluta æqualis eſt numero eorundem in quantitate reſolvenda. Reducatur utraque quantitas ad Poteſtates Indeterminatæ, factâ multiplicatione, & habebitur

$$z^4 + 2z^3 - 11z^2 - 12z = az^4 \left.\begin{matrix} -6a \\ +\;\; b \end{matrix}\right\} z^3 \left.\begin{matrix} +11a \\ -3b \\ +c \end{matrix}\right\} z^2 \left.\begin{matrix} -6a \\ +2b \\ -c \\ +d \end{matrix}\right\} z.$$

Atque comparando Terminos homologos, obtinebimus $a = 1$, $b - 6a = 2$, $c - 3b + 11a = -11$, $d - c + 2b - 6a = -12$; ex quibus eruitur $a = 1$, $b = 8$ $c = 2$, $d = -20$; hinc fit quantitas propoſita

$$z^4 + 2z^3 - 11z^2 - 12z = z.\overline{z-1}.\overline{z-2}.\overline{z-3} + 8z.\overline{z-1}.\overline{z-2} + 2z.\overline{z-1} - 20z.$$

Et ad eundem prorſus modum procedere licet in aliis caſibus. Brevitatis autem gratiâ, accipe regulam ſequentem. Divide unitatem per Terminos hujus progreſſionis continuè, $n-1$, $n-2$, $n-3$, $n-4$, &c. hoc eſt, divide unitatem per $n-1$, & Quotum prodeuntem per $n-2$, & noviſſimum Quotum per $n-3$, & ſic porro. Tum Quotos omnes ſic prodeuntes diſpone regulariter in Tabula ut vides, rejectis Dignitatibus ipſius n, & retentis ſolummodo Coefficientibus, utpote qui ſoli huic negotio ſunt utiles; atque habebis

Tabulam

Tabulam priorem.

1	1	1	1	1	1	1	1	1	&c.
	1	3	7	15	31	63	127	255	&c.
		1	6	25	90	301	966	3025	&c.
			1	10	65	350	1701	7770	&c.
				1	15	140	1050	6951	&c.
					1	21	266	2646	&c.
						1	28	461	&c.
							1	36	&c.
								1	&c.
									&c.

Aſſume jam pro Coefficientibus, numeros in columnis deſcendentibus, & habebis valores Dignitatum ſequentes.

$z = z,$

$z^2 = z + z.\overline{z-1},$

$z^3 = z + 3z.\overline{z-1} + z.\overline{z-1}.\overline{z-2},$

$z^4 = z + 7z.\overline{z-1} + z.\overline{z-1}.\overline{z-2} + z.\overline{z-1}.\overline{z-2}.\overline{z-3},$

$z^5 = z + 15.z.\overline{z-1} + 25z.\overline{z-1}.\overline{z-2} + 10z.\overline{z-1}.\overline{z-2}.\overline{z-3} + z.\overline{z-1}.\overline{z-2}.\overline{z-3}.\overline{z+4},$ &c.

Hac itaque Tabulâ ſemel habitâ, quantitas quælibet reducitur ad formam quæſitam abſque tædio computi. Proponatur ea hactenus reducta $z^4 + 2z^3 - 11z^2 - 12z$. Excerpe valores Dignitatum ex Tabula, eoſque ducito reſpective in ſuos Coefficientes $-12, -11, +2,$ & 1, atque obtinebis

$$-12z = -12z,$$
$$-11z^2 = -11z - 11z.\overline{z-1},$$
$$+2z^3 = +2z + 6z.\overline{z-1} + 2z.\overline{z-1}.\overline{z-2},$$
$$+z^4 = +z + 7z.\overline{z-1} + 6z.\overline{z-1}.\overline{z-2} + z.\overline{z-1}.\overline{z-2}.\overline{z-3},$$
$$z^4 + 2z^3 - 11z^2 - 12z = -20z + 2z.\overline{z-1} + 8z.\overline{z-1}.\overline{z-2} + z.\overline{z-1}.\overline{z-2}.\overline{z-3}.$$

Et valores membrorum in unam ſummam collecti dant valorem totius ut jam prodiit. Notandum Seriem infinitam conflatam ex Dignitatibus

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tatibus Indeterminatæ aſcendentibus reduci non poſſe generaliter in aliam prædictæ formæ: nam quiſque Coefficiens eſſet Series infinita. In Seriebus autem finitis, res ſuccedit ut ſupra oſtenſum eſt.

Series etiam alterius formæ ſimiliter reducuntur. Fingamus enim eſſe quantitatem quamvis quæſitam

$$T=\frac{A}{z}+\frac{B}{z.\overline{z+1}}+\frac{C}{z.\overline{z+1}.\overline{z+2}}+\frac{D}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\&c.$$

Dein ſi occaſio poſtulet inquirere valorem ipſius T ſucceſſivum, ſcribe $z+1$ pro z, & emerget valor ſucceſſivus

$$T'=\frac{A}{\overline{z+1}}+\frac{B}{\overline{z+1}.\overline{z+2}}+\frac{C}{\overline{z+1}.\overline{z+2}.\overline{z+3}}+\frac{D}{\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}}+\&c.$$

Et ut hæc Series reducatur ad formam prioris, operor ad modum ſequentem,

$$\frac{A}{\overline{z+1}}\;.\;.\;.\;.\;.\;.=\frac{A}{z}-\frac{A}{z.\overline{z+1}},$$

$$\frac{B}{\overline{z+1}.\overline{z+2}}\;.\;.\;.=*\;.\;\frac{B}{z.\overline{z+1}}-\frac{2B}{z.\overline{z+1}.\overline{z+2}},$$

$$\frac{C}{\overline{z+1}.\overline{z+2}.\overline{z+3}}\;.\;.=*\;.\;.\;*\;.\;.\;\frac{C}{z.\overline{z+1}.\overline{z+2}}-\frac{3C}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}},$$

$$\frac{D}{\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}}=*\;.\;.\;*\;.\;.\;.\;.\;.\;*\;.\;.\;.\;\frac{D}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+$$

&c.

Et habeo $T'=\frac{A}{z}+\frac{B-A}{z.\overline{z+1}}+\frac{C-2B}{z.\overline{z+1}.\overline{z+2}}+\frac{D-3C}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\&c.$ Ubi nunc Denominatores iidem ſunt atque in valore ipſius T; eaque de cauſa, inſtituere licet comparationem Terminorum prout occaſio poſtulat. Hujuſmodi autem Operationes ſic demonſtrantur. Pone $\frac{1}{\overline{z+1}}=\frac{1}{z}-\frac{a}{z.\overline{z+1}}$, exiſtente a quantitate ſtatim inveniendâ; tum ducendo in Denominatorem $z.\overline{z+1}$, proveniet $z=z+1-a$, ſive delendo utrobique z, $0=1-a$, & $a=1$; quare ſubſtituendo unitatem pro a, habebitur $\frac{1}{\overline{z+1}}=\frac{1}{z}-\frac{1}{z.\overline{z+1}}$. Similiter fingo $\frac{1}{\overline{z+1}.\overline{z+2}}=\frac{1}{z.\overline{z+1}}-\frac{a}{z.\overline{z+1}.\overline{z+2}}$, & ducendo in Denominatorem, erit $z=z+$

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$2 - a$, ſive $a = 2$, atque exinde $\frac{1}{z+1.z+2} = \frac{1}{z.z+1} - \frac{2}{z.z+1.z+2}$.

Proponatur jam

$$\frac{A}{z+2} + \frac{B}{z+2.z+3} + \frac{C}{z+2.z+3.z+4} + \frac{D}{z+2.z+3.z+4.z+5} + \&c.$$

Quam reducere oportet in aliam debitæ formæ. Inſtituatur operatio ut mox oſtenſum, & invenies

$$\frac{A}{z+1} \text{ - - - - - - - } = \frac{A}{z} - \frac{2A}{z.z+1} + \frac{2A}{z.z+1.z+2},$$

$$\frac{B}{z+2.z+3} \text{ - - - - - } = * - \frac{B}{z.z+1} - \frac{4B}{z.z+1.z+2} + \frac{6A}{z.z+1.z+2.z+3},$$

$$\frac{C}{z+2.z+3.z+4} \text{ - - } = * \text{ - - - } * \text{ - - } \frac{C}{z.z+2.z+3} - \frac{6C}{z.z+1.z+2.z+3} + \&c.$$

$$\frac{D}{z+2.z+3.z+4.z+5} = * \text{ - - - } * \text{ - - - - } * \text{ - - - - } \frac{D}{z.z+1.z+2.z+3} - \&c.$$

Atque Series propoſita, ſub debita forma invenietur

$$\frac{A}{z} + \frac{B - 2A}{z.z+1} + \frac{C - 4B + 2A}{z.z+1.z+2} + \frac{D - 6C + 6B}{z.z+1.z+2.z+3} + \&c.$$

Et hunc in modum inſtituetur reductio in aliis caſibus. Si fractio reducenda ſit $\frac{1}{z+1}$, erunt duo membra in ejuſdem valore, ut in Exemplo ſuperiore. Si ſit $\frac{1}{z+2}$, erunt tria, ut in Exemplo poſteriore. Et in genere in valore ipſius $\frac{1}{z+n}$ ad debitam formam reducti, numerus membrorum excedet numerum n unitate. Hic tamen ſuppono n eſſe integrum & affirmativum; nam ſi ſit fractus vel negativus, valor fractionis $\frac{1}{z+n}$ excurret in infinitum.

Regula autem generalis pro hujuſmodi tranſmutationibus ea eſt quæ ſequitur. Duc Terminos hujus Progreſſionis n, $1+n$, $2+n$, $3+n$, &c. in ſe continuò, & Facta diſponantur in Tabula ſequente pro ratione Dignitatum numeri n, Coefficientibus tantùm reſervatis, & emerget

Tabula

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Tabula poſterior.

1									
1	1								
2	3	1							
6	11	6	1						
24	50	35	10	1					
120	274	225	85	15	1				
720	1764	1624	735	175	21	1			
5040	13068	13132	6769	1960	322	28	1		
40320	109584	105056	67284	22449	4536	546	36	1	
&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.

Deinde ſumendo Coefficientes ex columnis deſcendentibus, obtinebis valores Dignitatum,

$$\frac{1}{z^2} = \frac{1}{z.\overline{z+1}} + \frac{1}{z.\overline{z+1}.\overline{z+2}} + \frac{2}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{6}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$$

$$\frac{1}{z^3} = \frac{1}{z.\overline{z+1}.\overline{z+2}} + \frac{3}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{11}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$$

$$\frac{1}{z^4} = \frac{1}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} \text{ in } 1 + \frac{6}{z+4} + \frac{35}{\overline{z+4}.\overline{z+5}} + \frac{225}{\overline{z+4}.\overline{z+5}.\overline{z+6}} + \&c.$$

Et ita porro in reliquis. Adeo ut habitâ Serie ex dignitatibus compoſitâ, ea ſemper reduci poſſit in aliam formæ deſideratæ, ope hujus Tabulæ.

Vel propoſitâ Serie $\frac{A}{z^2} + \frac{B}{z^3} + \frac{C}{z^4} + \frac{E}{z^5} + \&c.$ ſume Coefficientes ex columnis tranſverſis, & ponantur

$a = A,$
$b = A + B,$
$c = 2A + 3B + C,$
$d = 6A + 11B + 6C + D,$
$e = 24A + 50B + 35C + 10D + E,$
$f = 120A + 274B + 225C + 85D + 15E + F,$
&c.

Atque

Atque Series ex Dignitatibus compoſita tranſmutabitur in ſequentem debitæ formæ

$$\frac{a}{z.\overline{z+1}}+\frac{b}{z.\overline{z+1}.\overline{z+2}}+\frac{c}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\frac{d}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}}+\&c.$$

Proponatur nunc Fractio $\frac{1}{zz+nz}$, primùm ope Diviſionis reſolvo eam in Seriem vulgarem $\frac{1}{z^2}-\frac{n}{z^3}+\frac{n^2}{z^4}-\frac{n^3}{z^5}+\frac{n^4}{z^6}-$ &c. unde eſt $A=1$, $B=-n$, $C=+n^2$, $D=-n^3$, $E=+n^4$, &c. & hiſce valoribus ſubſtitutis, emergent $a=1$, $b=1-n$, $c=2-3n+n^2$, $d=6-11n+6n^2-n^3$, &c. atque adeo

$$\frac{1}{zz+nz}=\frac{1}{z.\overline{z+1}}+\frac{1-n}{z.\overline{z+1}.\overline{z+2}}+\frac{2-3n+n^2}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\frac{6-11n+6n^2-n^3}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}}+\&c.$$

Hoc eſt $\frac{1}{zz+nz}=\frac{1}{z.\overline{z+1}}+\frac{1-n}{z+2}A+\frac{2-n}{z+3}B+\frac{3-n}{z+4}C+\frac{4-n}{z+5}D+$ &c.

Ubi quantitates A, B, C, D, &c. jam deſignant Terminos hujus Seriei more *Newtoniano*. Et patet Seriem abrumpere quotieſcunque eſt n integer & affirmativus. In aliis Exemplis denotet etiam z minimum Factorum in Denominatore, & Series ſemper abrumpet hac methodo ubi ejus natura tulerit. Ut ſi Fractio ſit $\frac{1}{x.\overline{x-3}.\overline{x+2}}$, pono $z=x-3$, minimo utique trium Factorum; tum erit $x=z+3$, & $x+2=z+5$: Atque evadet Fractio $\frac{1}{z.\overline{z+3}.\overline{z+5}}$, ſive multiplicatione facta, $\frac{1}{z^3+8z^2+15z}$, quæ dividendo fit $\frac{1}{z^3}-\frac{8}{z^4}+\frac{49}{z^5}-\frac{272}{z^6}+\frac{1441}{z^7}-\frac{7448}{z^8}+\frac{37969}{z^9}-$ &c. Unde eſt $A=0$, $B=1$, $C=-8$, $D=+49$, $E=-272$, $F=+1441$, $G=-7448$, &c. Et exinde prodeunt $a=0$, $b=1$, $c=-5$, $d=12$, $e=-12$; at f & reliquæ ſunt nihil: & per conſequens abrumpit Series, exiſtente accuratè

$$\frac{1}{z.\overline{z+3}.\overline{z+5}}=\frac{1}{z.\overline{z+1}.\overline{z+2}}\text{ in }1-\frac{5}{z+3}+\frac{12}{\overline{z+3}.\overline{z+4}}-\frac{12}{\overline{z+3}.\overline{z+4}.\overline{z+5}}.$$

In fractione quavis $\frac{1}{z.\overline{z+a}.\overline{z+b}.\overline{z+c},\&c.}$ ſit z minimus Factorum, adeo, ut a, b, c, &c. ſint affirmativi, & ſi ſint quoque integri, Series terminabitur, aliàs excurret in infinitum. Ubi vero Series abrumpit, eadem inveniri poteſt plurimis modis idquè elegantiùs quàm per regulam ſuperiorem generalem: quippe a concinnitate alieniſſimum eſt, primo

primo reducere Fractionem finitam in Seriem infinitam, ut poſtea ejus valor habeatur in Terminis numero finitis: quod nos hic fecimus ut regula generalis illuſtraretur, non ut doceretur optima methodus quando Series abrumpit.

Si in Tabula priore excerpantur numeri ex columnis aſcendentibus, & ponantur

$$a = A,$$
$$b = B - A,$$
$$c = C - 3B + A,$$
$$d = D - 6C + 7B - A,$$
$$e = E - 10D + 25C - 15B + A,$$
$$f = F - 15E + 65D - 90C + 31B - A,$$
&c.

Tum Series hujus formæ $\frac{A}{z.\overline{z+1}} + \frac{B}{z.\overline{z+1}.\overline{z+2}} + \frac{C}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} +$ &c.

migrabit in hanc $\frac{a}{z^2} + \frac{b}{z^3} + \frac{c}{z^4} + \frac{d}{z^5} +$ &c. quæ ex Dignitatibus componitur.

In hiſce Tranſmutationibus nullam habuimus rationem Termini $\frac{A}{z}$, quoniam is abſque ulla Tranſmutatione ambiguè pertinet tam ad Seriem Dignitatum quam ad eam Factorum.

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PARS PRIMA.

DE

Summatione Serierum.

IN hac parte prima conatus ſum abbreviare computos in Quadratura Curvarum, & in Problematibus etiam difficilioribus, idque perveniendo ad valores Serierum infinitarum magis expeditè quàm per ſimplicem additionem Terminorum ut vulgò fit. Hæc quidem in citò convergentibus abunde rem conficit, neque ulteriore opus eſt artificio: attamen ubi lentè convergunt immenſus plerumque labor requiritur, & eo quidem major quo minor eſt convergentia; quod ſi lentiſſimè approximent, evadunt penitus intractabiles. Etenim notiſſimum eſt nonnunquam opus eſſe ultra mille numero Terminos, ut ſumma habeatur juſta ad duas treſve figuras. Monſtrabimus ergo in ſequentibus methodum expeditam tranſmutandi eas omnium lentiſſime convergentes in alias celerrime approximantes; ex quibus ſcilicet ſummæ ſupputari queunt minimo labore ad plurima figurarum loca.

Abrumpent quidem Series tranſmutatæ ubi ſummandæ ſunt ſummabiles; eoque in caſu tranſmutatio evadet ſummatio. Sed de Seriebus ſummabilibus minus ſollicitus ſum eaſque obiter tantum attingo, utpote quæ in Quadraturis raro prodire ſolent. Hic enim non operam collocavi in eruendis Seriebus inutilibus quæ per obvia Theoremata ſunt

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ſunt ſummabiles, ſed in eruendis Theorematibus per quæ Series utiles expedite poſſint ſummari ad tot figurarum loca quot uſus quicunque requirunt.

De Seriebus ſimplicioribus.

Non tantum Seriei convergentia ſed & ſimplicitas plurimum confert ad contrahendos calculos. Quapropter priuſquam ad Tranſmutationes accedamus, ſciendum eſt Series *Newtonianas* in Tractatu de Quadratura Curvarum, non ſolùm abrumpere ubi rei natura tulerit, ſed & eſſe omnium ſimpliciſſimas, ubi excurrunt in infinitum, atque eapropter præferendas eſſe iis quæ inveniuntur methodo vulgari, ſcilicet reducendo Ordinatas in Series convergentes, ut exinde computentur Areæ.

Sit $x^{\theta-1}\times\overline{e+fx^{\eta}}^{\,\lambda-1}$ Ordinata Curvæ, in qua x eſt Abſciſſa, e & f Coefficientes, atque $\theta-1$, $\lambda-1$, & η indices Dignitatum: pone $r=\frac{\theta+\eta}{\eta}$, $s=\frac{\theta+\lambda\eta}{\eta}$, & juxta *Newtonum* Area erit

$$\frac{x^{\theta}}{\theta e}\times\overline{e+fx^{\eta}}^{\,\lambda}-\frac{s}{r}A\frac{fx^{\eta}}{e}-\frac{s+1}{r+1}B\frac{fx^{\eta}}{e}-\frac{s+2}{r+2}C\frac{fx^{\eta}}{e}-\frac{s+3}{r+3}D\frac{fx^{\eta}}{e}-\&c.$$

Ubi A, B, C, D, &c. deſignant Terminos, quemque in ſuo ordine ab initio: ſcilicet eſt $A=\frac{x^{\theta}}{\theta e}\times\overline{e+fx^{\eta}}^{\,\lambda}$, $B=-\frac{s}{r}A\frac{fx^{\eta}}{e}$, $C=-\frac{s+1}{r+1}B\frac{fx^{\eta}}{e}$, & ſic deinceps. Proponatur jam inventio Arcûs ex dato Sinu recto x, vel quod perinde eſt, Quadratura Curvæ cujus Ordinata eſt $\frac{1}{\sqrt{1-xx}}$: hæc ad debitam formam reducta fit $x^{0}\times\overline{1-x^{2}}^{\,-\frac{1}{2}}$, quæ comparata cum Ordinata generali dat $e=1$, $f=-1$, $\eta=2$, $\theta-1=0$, $\lambda-1=-\frac{1}{2}$; adeoque $\theta=1$, $\lambda=\frac{1}{2}$: et exinde $r=\frac{3}{2}$, $s=1$; quibus valoribus in Theoremate ſubſtitutis, oritur pro Arcu Series

$$x\sqrt{1-xx}+\frac{2}{3}Ax^{2}+\frac{4}{5}Bx^{2}+\frac{6}{7}Cx^{2}+\frac{8}{9}Dx^{2}+\frac{10}{11}Ex^{2}+\&c.$$

At ſi Ordinata propoſita prius reſolvatur in Seriem per Theorema *Newtoni* pro evolvendo Binomio, dein capiatur Fluens cujuſque Termini, prodibit pro eodem Arcu Series

$$x+\frac{1\times1}{2\times3}Ax^{2}+\frac{3\times3}{4\times5}Bx^{2}+\frac{5\times5}{6\times7}Cx^{2}+\frac{7\times7}{8\times9}Dx^{2}+\frac{9\times9}{10\times11}Ex^{2}+\&c.$$

Unde patet priorem longe ſimpliciorem eſſe & proinde facilius continuari in infinitum. Exempli gratiâ, ſi Arcus quæſitus ſit octava pars totius Circum-

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Circumferentiæ, ejus Sinus x erit æqualis $\surd\frac{1}{2}$, quo ſubſtituto, evadet Series

Prior $\frac{1}{2} + \frac{1}{3}A + \frac{2}{5}B + \frac{3}{7}C + \frac{4}{9}D + \frac{5}{11}E +$ &c.

Poſterior $\surd\frac{1}{2} + \frac{1}{12}A + \frac{9}{40}B + \frac{25}{84}C + \frac{49}{144}D + \frac{81}{220}E +$ &c.

Quo in caſu prior duplici de cauſa eſt præferenda, tum quoniam per Factores ſimpliciores producitur, tum quia libera eſt a numero ſurdo qui reperitur in poſteriore. Attamen ubi x eſt quantitas rationalis, & ſimul $x\sqrt{1-xx}$ irrationalis, Series poſterior eligenda eſt, modo ſit x magnitudinis uſque adeo contemnendæ quæ efficiat Seriem celerrime convergentem; hac enim ratione evitatur extractio Radicis quadratæ. Inſuper ſi ſit $x = 1$, neceſſe habemus recurrere ad ſecundam, quoniam in illo caſu evaneſcit quantitas $\sqrt{1-xx}$, in quam prima multiplicatur.

De Seriebus quæ celerius convergunt.

Ubi quantitas indeterminata, creſcente eâ quæ deſideratur, cito evadit prægrandis, & tandem infinite magna; Termini Seriei ex illa compoſitæ erunt alternatim negativi & affirmativi, & approximabunt lentius quam ubi indeterminata ultra datam magnitudinem creſcere nequit. Ut ſi quæratur Area vel Arcus circularis, melius eſt adhibere Sinum rectum qui nequit eſſe major Radio, quam Tangentem quæ cito excreſcit in immenſam longitudinem; ut olim obſervavit *Newtonus.* Et e contra præferendæ ſunt Tangentes in Hyperbola, utpote quæ datam magnitudinem ſuperare nequeunt, ſed continentur intra determinatos limites, eoſque ſatis arctos. Sed quæ hic diximus, non impediunt quo minus Area vel Arcus, magnitudinis mediocris vel contemnendæ, poſſit exquiri prout cuique viſum fuerit: nam differentia tantùm eſt notabilis in iis caſibus in quibus quantitates quæſitæ ſunt magnæ. Atque Series quarum Termini ſunt per vices negativi & affirmativi ſunt magis tractabiles quam alteræ, ubi de Summatione agitur. Quæ autem hic dicta ſunt de Curvis binomialibus, obtinent quoque in iis ſuperiorum nominum.

Verum quidem eſt Series celeriter convergentes multifariam inveniri poſſe, adhibita Methodo differentiali *Newtoniana.* Sed quo magis convergunt eò magis ſolent eſſe compoſitæ: quapropter præfero ſimpliciores etiamſi lentius convergentes.

De Summis ſucceſſivis.

Per Summam ſucceſſivam intelligo quantitatem quæ ſuccedit Summæ omnium Terminorum, quando Termini ſubſequentes deveniunt in loca antecedentium. Ut ſi Summa ſit $T+T'+T''+T'''+T^{iv}+T^{v}+$ &c. ſcribe Terminos poſteriores pro prioribus, & habebis Summam ſucceſſivam $T'+T''+T'''+T^{iv}\ T^{v}$ &c. in qua denuo ſi ſubſtituantur Termini conſequentes pro antecedentibus, proveniet Summa $T''+T'''+T^{iv}+T^{v}+T^{vi}+$ &c. quæ ſuccedit noviſſimæ; & ſic deinceps.

Hinc ſi S, S', S'', S''', &c. denotent Summas ſucceſſivas, erunt

$$S = T+T'+T''+T'''+T^{iv}+ \text{\&c.}$$
$$S' = \text{- -}\ T'+T''+T'''+T^{iv}+ \text{\&c.}$$
$$S'' = \text{- - - - -}\ T''+T'''+T^{iv}+ \text{\&c.}$$
$$S''' = \text{- - - - - - - -}\ T'''+T^{iv}+ \text{\&c.}$$

Hoc eſt, ſi a Serie quavis infinita ſubducatur primus Terminus, & a reſidua ſubducatur etiam primus Terminus, dein ab ea quæ relinquitur rurſus ſubducatur primus Terminus, & ſic ad libitum; Series hoc modo gradatim mulctatæ primis Terminis, erunt Summæ ſucceſſivæ; hoc eſt,

$$S'=S-T,\quad S''=S'-T',\quad S'''=S''-T'',\ \text{\&c.}$$

$$\text{Vel } S' = S-T,$$
$$S'' = S-T-T',$$
$$S''' = S-T-T'-T'',$$
$$S^{iv} = S-T-T'-T''=T''',$$
&c.

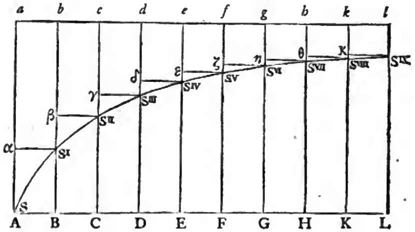
Hic locuti ſumus de Summis Terminorum omnium in infinitum, quæ incipiunt ad datum quemvis Terminum; nam quicunque Terminus ſit T, S erit Summa ejus & omnium ſequentium, item S' erit Summa ipſius T' & omnium reliquorum. Hæc quidem obtinent ubi agitur de Summa Terminorum numero infinitorum: Attamen ubi agitur de Summa Terminorum quorum numerus eſt finitus, S erit Summa Terminorum omnium abinitio ad datum quemvis Terminum T, & S' erit Summa eorundem Terminorum dempto T, atque S'' erit Summa eorundem demptis duobus T, T', & ſic in reliquis.

Hinc in Summatione Terminorum a dato quovis Termino ad infinitum, ſi z ſit longitudo Abſciſſæ quæ correſpondit Summæ S; tum $z+1$, $z+2$, $z+3$, &c. erunt longitudines ejuſdem reſpective correſpondentes Summis ſucceſſivis S', S'', S''', &c. Et e contra in Summatione

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matione Terminorum a dato quovis uſque ad initium Seriei, longitudines $z-1$, $z-2$, $z-3$, &c. reſpondebunt Summis S′, S″, S‴, &c. dummodo ipſi S reſpondeat Abſciſſa *z*. Nam in primo caſu diſtantiæ Summarum ab initio perpetuo increſcunt incremento Abſciſſæ, decreſcuntque eodem decremento in poſteriore.

Sit S S′ Curva quævis cujus Aſymptotos eſt *a b*, eique parallela Abſciſſa A B. Dividatur Abſciſſa in partes innumeras inter ſe æquales A B, B C, C D, &c. Et a punctis diviſionum A, B, C, D, &c. erigantur perpendicula ad Aſymptoton, decuſſantia Curvam in punctis S, S′, S″, &c. Aſymptoton vero in *a*, *b*, *c*, &c. A punctis S′, S″, S‴, &c. ad Ordi-



natas proxime antecedentes ducantur S′α, S″β, S‴γ, $S^{iv}\delta$, &c. parallelæ Abſciſſæ; adeo ut Sα, S′β, S″γ, S‴δ, &c. ſint differentiæ Ordinatarum tam earum quæ extenduntur a Curva ad Aſymptoton, quam illarum quæ a Curva ad Abſciſſam porriguntur. Igitur Ordinatæ interceptæ inter Curvam & Aſymptoton exponent Summas, & differentiæ pergendo ab iiſdem in infinitum exponent Terminos. Hoc eſt, ſi *e* S^{iv} deſignet Summam, ſuccedentes erunt *f* S^{v}, *g* S^{vi}, *h* S^{vii}, &c. quarum differentiæ ε S^{iv}, ζ S^{v}, η S^{vi}, &c. in infinitum ſunt Termini quorum Summa eſt *e* S^{iv}. Et ſimiliter ſi E S^{iv}, D S‴, C S″, &c. deſignent Summas ſucceſſivas, quarum prima eſt E S^{iv}, differentiæ antecedentes δ S‴, γ S″, &c. exponent Terminos numero finitos pergentes ab Ordinata E S^{iv} uſque ad initium Seriei. Summatio igitur Serierum redu-

reducitur ad inventionem Ordinatarum ex datis earum Differentiis. Sed notandum oportere Summam ultimam eſſe nihil in utroque caſu; quod ſemper fiet quando Curva tranſit per punctum A in Abſciſſa, & ſimul habet *ab* pro Aſymptoto. Hæc cautio adhibenda eſt ut Summæ inveſtigandæ per methodos tradendas, ſint veræ, correctione minime indigentes, ut ſæpiſſime fit in quadratura Curvarum.

Propositio I.

Si Termini alicujus Seriei formentur ſcribendo numeros 1, 2, 3, 4, 5, *&c. pro* z, *in Quantitate* $A + Bz + Cz.\overline{z-1} + Dz.\overline{z-1}.\overline{z-2} + Ez.\overline{z-1}.\overline{z-2}.\overline{z-3}$ + *&c. tum Summa Terminorum ab initio quorum numerus eſt* z, *erit*

$Az + \overline{z+1}$ in $\frac{1}{2} Bz + \frac{1}{3} Cz.\overline{z-1} + \frac{1}{4} Dz.\overline{z-1}.\overline{z-2} + \frac{1}{5} Ez.\overline{z-1}.\overline{z-2}.\overline{z-3}$ + &c.

Ubi notandum eſt quantitatem $z+1$ duci in totam Seriem quæ eam immediate ſequitur. Propoſitio autem ſic demonſtratur. Fingè Summam $S = Az + \overline{z+1}$ in $\frac{1}{2} Bz + \frac{1}{3} Cz.\overline{z-1} + \frac{1}{4} Dz.\overline{z-1}.\overline{z-2}$ + &c. ſive $S = Az + \frac{1}{2} B\overline{z+1}.z + \frac{1}{3} C\overline{z+1}.z.\overline{z-1} + \frac{1}{4} D\overline{z+1}.z.\overline{z-1}.\overline{z-2}$ + &c. Dein ſcribe valores variabilium ſuccedentes pro preſentibus; hoc eſt, $S-T$ pro S; & $z-1$ pro z; atque obtinebis $S-T = A\overline{z-1} + \frac{1}{2} Bz.\overline{z-1}. + \frac{1}{3} Cz.\overline{z-1}.\overline{z-2} + \frac{1}{4} Dz.\overline{z-1}.\overline{z-2}.\overline{z-3}$ + &c. ſubducito nunc hanc Æquationem de priore, & manebit $T = A + Bz + Cz.\overline{z-1} + Dz.\overline{z-1}.\overline{z-2}$ + &c. Unde e contra ſi detur hic valor Termini ut in Propoſitione, Summa erit ea quæ aſſignata eſt. Inſuper hæc Summa evadit nihil quando eſt z nihil: adeoque conſtat Theorema. Q. E. D.

Exemplum I.

Detur Series numerorum naturalium 1, 2, 3, 4, 5, 6, &c. hi formantur ſcribendo 1, 2, 3, &c. pro z, in ipſa quantitate z, quam itaque comparans cum Termino in Theoremate, erit $A = 0$, $B = 1$, & C, D, E, & ſequentes nihil; quibus valoribus ſubſtitutis prodit Summa $\overline{z+1}$ in $\frac{1}{2} z$, ſive $\frac{zz+z}{2}$ pro aggregato tot Terminorum Seriei propo-

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propoſitæ quot ſunt unitates in z. Ut ſi ſit $z=6$, proveniet $\frac{36+6}{2}=21$ pro Summa primorum ſex Terminorum.

Exemplum II.

Detur jam Series numerorum imparium 1, 3, 5, 7, 9, 11, &c. hi formantur ſcribendo 1, 2, 3, 4, &c. in Quantitate $2z-1$, id eſt, $-1+2z$, quæ comparata cum valore Termini generali, dat $A=-1$, $B=2$, C, D, E, &c. nihil; quibus ſcriptis in Summa, emergit $-z+\overline{z+1}$ in $\frac{2z}{2}$, vel zz pro aggregato tot Terminorum quot numerat z. Atque ita quidem ſe res habet in caſu præſente, nam Summæ ſucceſſivæ ſunt Quadrata numerorum naturalium.

Exemplum III.

Summanda ſit Series Quadratorum 1, 4, 9, 16, 25, 36, 49, &c. quæ formantur expreſſione zz. Per ea quæ in Introductione explicantur, quantitas zz reducta ad formam Theorematis, evadit $z+z.\overline{z-1}$; adeoque $A=0$, $B=1$, $C=1$, & inde fit Summa $\overline{z+1}$ in $\frac{z}{2}+\frac{z.\overline{z-1}}{3}$, hoc eſt $\frac{z.\overline{z+1}.\overline{2z+1}}{6}$. Exempli cauſâ, ſcribe 7 pro z, & habebis $\frac{7.8.15}{6}=140$, quod eſt aggregatum ſeptem Terminorum.

Exemplum IV.

Proponantur nunc Quadrata numerorum imparium 1, 9, 25, 49, 81, 121, 169, &c. eadem formantur ſcribendo 1, 2, 3, 4, &c. ſucceſſive in expreſſione $1+4zz-4z$, quæ ſic ſcripta $1+4z.\overline{z-1}$, dat $A=1$, $B=0$, $C=4$, D, E, &c. nihil: & hiſce ſubſtitutis, prodit Summa $z+\overline{z+1}$ in $\frac{4}{3}z.\overline{z-1}$, ſive $\frac{4z^3-z}{3}$.

Exemplum V.

Si dentur Cubi 1, 8, 27, 64, 125, 216, &c. quos aſſignat z^3, reducatur z^3 ad debitam formam $z+3z.\overline{z-1}+z.\overline{z-1}.\overline{z-2}$; eritque $A=0$, $B=1$, $C=3$, $D=1$, reliquæque erunt nihil; & propterea

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eſt Summa $\overline{z+1}$ in $\frac{1}{2}z+z.\overline{z-1}+\frac{1}{4}z.\overline{z-1}.\overline{z-2}$, quæ concinnata evadit $\frac{z^2}{4}\times\overline{z+1}^2$. Et hinc conſtat Summas horum Cuborum eſſe Quadrata numerorum 1, 3, 6, 10, 15, &c. ſcilicet Triangularium.

Scholion.

Hujuſmodi Series facilius ſummantur per Terminorum Differentias: deſignent enim A, A2, A3, &c. Seriem ſummandam; collige Terminorum differentias primas B, B2, B3, &c. ſecundas C, C2, C3, &c. tertias D, D2, &c. & ſic porro uſque dum perventum fuerit ad ultimam quæ hic eſt E: & Summa Terminorum cujus numerus eſt z, erit

A A2 A3 A4 A5
B B2 B3 B4
C C2 C3
D D2
E

$$A\frac{z}{1}+B\frac{z}{1}\times\frac{z-1}{2}+C\frac{z}{1}\times\frac{z-1}{2}\times\frac{z-2}{3}+D\frac{z}{1}\times\frac{z-1}{2}\times\frac{z-2}{3}\times\frac{z-3}{4}+\text{\&c.}$$

Sed notandum eſt quod differentiæ ſumi debeant auferendo priores de poſterioribus, hoc eſt, ponendo $B=A2-A$, $B2=A3-A2$, &c. tum $C=B2-B$, &c. Hujus autem demonſtratio pendet ex methodo differentiali *Newtonianâ*.

Proponantur Series 1, —1, 0, 8, 27, 61, 114, 119, &c. & collectis differentiis ad modum ſupra expoſitum, invenietur $A=1$, $B=-2$, $C=3$, $D=4$, reliquæ autem nihil ſunt, adeoque evadit ſumma

1, —1, 0, 8, 27, 61, 114, 190,
—2, 1, 8, 19, 34, 53, 76,
3, 7, 11, 15, 19, 23,
4, 4, 4, 4, 4,

$$\frac{z}{1}-2\times\frac{z}{1}\times\frac{z-1}{2}+3\times\frac{z}{1}\times\frac{z-1}{2}\times\frac{z-2}{3}+4\times\frac{z}{1}\times\frac{z-1}{2}\times\frac{z-2}{3}\times\frac{z-3}{4},$$

quæ in ordinem redacta fit $\frac{\overline{z-3}.z-2.z.z+2}{6}$. Series autem formatur ſcribendo 0, 1, 2, 3, 4, &c. in quantitate $\frac{4z^3-3zz-13z+6}{6}$.

Propo-

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Propositio II.

Si Termini cujuſcunque Seriei formentur ſcribendo numeros quoſvis unitate differentes in Quantitate

$$\frac{A}{z.z+1}+\frac{B}{z.z+1.z+2}+\frac{C}{z.z+1.z+2.z+3}+\frac{D}{z.z+1.z+2.z+3.z+4}+$$

&c. Summa omnium Terminorum in infinitum, incipiens ad datum quemlibet Terminum erit

$$\frac{A}{z}+\frac{B}{2z.z+1}+\frac{C}{3z.z+1.z+2}+\frac{D}{4z.z+1.z+2.z+3}+\text{\&c.}$$

Pone Summam S =

$$\frac{A}{z}+\frac{B}{2z.z+1}+\frac{C}{3z.z+1.z+2}+\frac{D}{4z.z+1.z+2.z+3}+\text{\&c.}$$

Tum ſcribe valores ipſorum S & z, conſequentes pro antecedentibus, hoc eſt S — T pro S, & $z+1$ pro z, quoniam jam agitur de Terminis numero infinitis: & proveniet $S-T=\frac{A}{z+1}+\frac{B}{2.z+1.z+2}+\frac{C}{3.z+1.z+2.z+3}+\frac{D}{4.z+1.z+2.z+3.z+4}+\text{\&c.}$ Hæc Æquatio ſubducta de ſuperiore, relinquit $T=\frac{A}{z.z+1}+\frac{B}{z.z+1.z+2}+\frac{C}{z.z+1.z+2.z+3}+\frac{D}{z.z+1.z+2.z+3.z+4}+\text{\&c.}$ Unde è contra ſi detur hic Terminus, Summa erit ea quæ in Propoſitione aſſignatur. Q. E. D.

Corollarium 1. Si Terminus ſit $\frac{p}{z.z+1.z+2.z+3.z+4.\text{\&c.}}$, rejice ultimum Factorem, dein divide reſiduum per numerum Factorum qui relinquuntur, atque habebis Summam Terminorum. Sit Terminus $\frac{A}{z.z+1}$; rejice ultimum Factorem $z+1$, & manebit $\frac{A}{z}$; quumque unicus ſit Factor reſiduus z, erit $\frac{A}{z}$ Summa omnium Terminorum.

Sit

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Sit jam Terminus conſtans ex tribus Factoribus $\frac{B}{z.\overline{z+1}.\overline{z+2}}$, rejice ultimum Factorem $z+2$, & reſtabit $\frac{B}{z.\overline{z+1}}$, quæ diviſa per 2, numerum utique Factorum qui relinquuntur, exhibet $\frac{B}{2z.\overline{z+1}}$ pro Summa.

Similiter ſi ex Termino $\frac{C}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}$ ex quatuor Factoribus conſtato, rejiciatur ultimus $z+3$, & reſiduum dividatur per 3, obtinebitur Summa $\frac{C}{3z.\overline{z+1}.\overline{z+2}}$.

Si Terminus ſit $\frac{A}{z}$, rejice Factorem z, & quoniam nullus reſtat, divide A per Nihil, & habebis pro Summâ Quantitatem infinite magnam, uti notum eſt. Et hac de re primus quod ſciam egit D. *Taylor* in Methodo Incrementorum. Eadem etiam fuſius & elegantiſſime traditur a D. *Nicol* in Actis Academiæ Regiæ Pariſienſis.

Corollarium 2. Per ea quæ hac ſuper materia habentur in Introductione, conſtat Terminum quempiam $\frac{A}{z.\overline{z+a}.\overline{z+b}.\overline{z+c}.\&c.}$ reſolvi ſemper poſſe in Terminos duos forte aut plures ſummabiles & numero finitos, quando a, b, c, &c. ſunt numeri integri; in illo igitur caſu Series ſummabitur. Ut ſi Terminus ſit $\frac{1}{z.\overline{z+3}}$, idem reſolvitur in tres Terminos ſummabiles $\frac{1}{z.\overline{z+1}}-\frac{2}{z.\overline{z+1}.\overline{z+2}}+\frac{2}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}$. Unde per Corollarium præcedens, erit Summa $\frac{1}{z}-\frac{1}{z.\overline{z+1}}+\frac{2}{3z.\overline{z+1}.\overline{z+2}}$, quæ ſimul juncta ſunt $\frac{3z^2+6z+2}{3z^3+9z^2+6z}$. Et ſimiliter ſi Terminus ſit hujus formæ $\frac{a+bz^2+cz^3+dz^4+\&c.}{z.\overline{z+a}.\overline{z+b}.\overline{z+c}.\overline{z+d}.\&c.}$ Series ſummabitur, dummodo a, b, c, d, &c. ſint integri, & numerus Factorum in Denominatore, ad minimum binario excedat altiſſimum ipſius z dimenſionem in Numeratore. Sed excipio caſus in quibus duo aut plures Factores in Denominatore ſibi invicem æquantur; in illis Series non ſunt ſummabiles.

Exemplum I.

Proponatur Series ſummanda $\frac{1}{1.4.7}+\frac{1}{4.7.10}+\frac{1}{7.10.13}+\frac{1}{10.13.16}+\frac{1}{13.16.19}$ + &c. Termini hujus Seriei aſſignantur quantitate $\frac{1}{3z.3z+3.3z+6}$, uti patebit ſcribendo pro z ſucceſſive $\frac{1}{3}$, $1\frac{1}{3}$, $2\frac{1}{3}$, $3\frac{1}{3}$, &c. id eſt, hâc $\frac{1}{27.z.z+1.z+2}$; unde fit Summa $\frac{1}{54z.z+1}$. Nam in eadem ſcribendo pro z valorem ſuum primum $\frac{1}{3}$, prodibit $\frac{1}{24}$ pro Summa totius Seriei. Si ſcribatur pro z valor ſuus ſecundus $1\frac{1}{3}$, prodibit $\frac{1}{168}$ pro Summa totius Seriei dempto primo Termino. Si pro z ſcribatur valor ſuus tertius $2\frac{1}{3}$, prodibit $\frac{1}{420}$ pro Summa totius Seriei demptis primis duobus Terminis. Et ſic in infinitum.

Exemplum II.

Proponatur Series $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}+\frac{1}{5.8}$ + &c. Termini hujus aſſignantur quantitate $\frac{1}{z.z+3}$, in qua oportet ſcribi 1, 2, 3, 4, &c. ſucceſſive pro z. Quantitas autem $\frac{1}{z.z+3}$ reducitur in tres Terminos ſummabiles nempe $\frac{1}{z.z+1}-\frac{2}{z.z+1.z+2}+\frac{2}{z.z+1.z+2.z+3}$; Unde erit Summa $\frac{1}{z}-\frac{1}{z.z+1}+\frac{2}{3z.z+1.z+2}$, ſive $\frac{3z^2+6z+2}{3z.z+1.z+2}$. Jam ſi vis aggregatum omnium omnino Terminorum, ſubſtitue in Summa unitatem pro z, & obtinebis $\frac{3+6+2}{3.1.2.3}$, id eſt $\frac{11}{18}$ pro valore Seriei propoſitæ.

Exemplum III.

Detur jam Series $\frac{1}{2.3.4.5}+\frac{4}{3.4.5.6}+\frac{9}{4.5.6.7}+\frac{16}{5.6.7.8}+\frac{25}{6.7.8.9}$ + &c. ubi Numeratores ſunt Quadrata numerorum naturalium;

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Terminus quilibet in genere aſſignabitur expreſſione $\frac{zz-2z+1}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}$, exiſtentibus 2, 3, 4, 5, &c. valoribus indeterminatæ ſucceſſivis. Et quantitas illa reſolvitur in tres Terminos ſummabiles ſcilicet $\frac{1}{z.\overline{z+1}} - \frac{7}{z.\overline{z+1}.\overline{z+2}} + \frac{16}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}$: & inde Summa $\frac{1}{z} - \frac{7}{2z.\overline{z+1}} + \frac{16}{3z.\overline{z+1}.\overline{z+2}}$, hoc eſt $\frac{6z^2-3z+2}{6.z.\overline{z+1}.\overline{z+2}}$, in qua ſi ſubſtituas 2 pro z, habebis $\frac{5}{36}$ pro valore Seriei.

Exemplum IV.

Quæratur valor Seriei $\frac{1}{1.2.3.4.5} + \frac{27}{2.3.4.5.6} + \frac{125}{3.4.5.6.7} + \frac{343}{4.5.6.7.8} +$ &c. ubi Numeratores ſunt Cubi numerorum imparium 1, 3, 5, 7, &c. Poſitis jam 1, 2, 3, 4, 5, &c. valoribus ipſius z ſucceſſivis, Termini aſſignabuntur per expreſſionem $\frac{8z^3-12z^2+6z-1}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}}$, quæ reſolvitur in $\frac{8}{z.\overline{z+1}} - \frac{84}{z.\overline{z+1}.\overline{z+2}} + \frac{386}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} - \frac{729}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}}$. Igitur eſt Summa $\frac{8}{z} - \frac{84}{2z.\overline{z+1}} + \frac{386}{3z.\overline{z+1}.\overline{z+2}} - \frac{729}{4z.\overline{z+1}.\overline{z+2}.\overline{z+3}}$, hoc eſt, $\frac{96z^3+72z^2+80z-3}{12z.\overline{z+1}.\overline{z+2}.\overline{z+3}}$; in qua ſi ſcribas primum valorem ipſius z, id eſt, unitatem pro z, invenies $\frac{245}{288}$ eſſe valorem Seriei.

Exemplum V.

Proponatur Series $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \frac{1}{9.10} +$ &c. quam invenit Vicecomes *Brounker* pro Quadratura Hyperbolæ; Terminus quivis in genere aſſignatur per expreſſionem $\frac{1}{4z.\overline{z+\frac{1}{2}}}$, ubi valores ipſius z ſunt $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, &c. Et quantitas reducta ad formam ſummabilem evadit

$$\frac{1}{4z.z+1}+\frac{1}{8z.z+1.z+2}+\frac{1.3}{16z.z+1.z+2.z+3}+\frac{1.3.5}{32z.z+1.z+2.z+3.z+4}+\&c.$$

Excurrit quippe in Seriem infinitam propterea quod differentia Factorum eſt numerus fractus in expreſſione aſſignante Terminos; hoc utique indicium eſt Seriem non eſſe ſummabilem. Regrediendo autem à Termino ad Summam, habebitur

$$\frac{1}{4z}+\frac{1}{16z.z+1}+\frac{1.3}{48z.z+1.z+2}+\frac{1.3.5}{128z.z+1.z+2.z+3}+\&c.$$

Quæ eſt Series eo citiùs convergens quo major eſt z. Sed ob faciliorem computum pone $A=\frac{1}{4z}$,

$$B=\frac{A}{2z+2},\ C=\frac{3B}{2z+4},\ D=\frac{5C}{2z+6},\ E=\frac{7D}{2z+8},\ F=\frac{9E}{2z+10},$$

&c. atque erit Summa, $A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\frac{1}{5}E+$ &c. In qua ſi pro z ſubſtituatur primus ejus valor $\frac{1}{2}$, habebitur valor totius Seriei ſummandæ: ſi pro z ſubſtituatur ejus valor ſecundus, proveniet Summa omnium Terminorum dempto primo; ſi pro z ſubſtituatur ejus valor tertius, proveniet Summa omnium exceptis primis duobus, & ſic porro. Subſtituo igitur pro z valorem ſuum decimum quartum $13\frac{1}{2}$, ut z ſit ſatis magna ad efficiendum Seriem celeriter convergere; & habeo $A=\frac{1}{54}$, $B=\frac{1}{29}A$, $C=\frac{3}{31}B$, $D=\frac{5}{33}C$, $E=\frac{7}{35}D$, $F=\frac{9}{37}E$, &c. quo in caſu Summa $A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+$ &c. æquabitur $\frac{1}{27.28}+\frac{1}{29.30}+\frac{1}{31.32}+$ &c. toti utique Seriei ſummandæ demptis primis tredecim Terminis. Horum igitur Summam quæro per additionem, invenioque fore .674285961. Dein ut obtineam Summam reliquorum, calculo eruo A, B, C, D, &c. ad tot decimalium loca quot eſt animus; eoſque inventos divido reſpective per 1, 2, 3, 4, 5, &c. ut vides.

A = .018518519	.018518519
B = - 638570	319285
C = - - 61797	20599
D = - - - 9363	2341
E = - - - 1873	375
F = - - - 455	76
G = - - - 128	18
H = - - - - 41	5
I = - - - - 14	1
	.018861219

Eoque

Eoque pacto obtineo .018861219 pro Summa Terminorum omnium poſt decimum tertium; hæc denique adjecta aggregato initialium prius invento, conficit .693147180 pro valore Seriei ſummandæ, id eſt, pro Logarithmo Hyperbolico binarii.

Quo plures Termini ſub initio colliguntur, eo citius converget Series quæ dat Summam reliquorum, propter z tanto majorem. Atque hujus methodi præſtantia in eo maxime enitet quod addendo Terminos aggregato initialium, z tot unitatibus augeatur, qua ratione Series tranſmutata ad libitum fere converget.

Quod autem in praxi impoſſibile ſit aſſequi Summas harum Serierum per meram collectionem Terminorum, patebit ex computo ſequente; ubi habetur Summa centum, mille, decies mille, & ſic porro uſque ad decies millies centena millia Terminorum.

Summa	100 - - - -	Terminorum	.690653446
	1000 - - - -		.692897242
	10000 - - -		.693122181
	100000 - - -		.693144680
	1000000 - -		.693146930
	10000000 - -		.693147155
	100000000 -		.693147178
	1000000000		.693147180

Ex hoc calculo conſtat centum Terminos dare Summam accuratam ad duas figuras; & gradatim decuplum Terminorum numerum colligendo, lucrari præterpropter unam tantummodo figuram: adeo ut ſi quis vellet eruere valorem hujus Seriei accuratum ad novem figurarum loca, nulla arte adhibita præter additionem, requirerentur circiter decies millies centena millia Terminorum. Et hæc Series convergit longe celerius quam aliæ quamplurimæ, quarum valores ſunt Quantitates finitæ.

Exemplum VI.

Summanda ſit Series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+$ &c. ubi Denominatores ſunt Quadrata numerorum 1, 2, 3, 4, &c. & Terminus in genere eſt $\frac{1}{zz}$: hic reductus in formam ſummabilem, evadit

$$\frac{1}{z.\overline{z+1}}+\frac{1}{z.\overline{z+1}.\overline{z+2}}+\frac{1.2}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\frac{1.2.3}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}}+\text{&c.}$$

Ergo Summma $=$

$$\frac{1}{z}+\frac{1}{2z.\overline{z+1}}+\frac{1.2}{3z.\overline{z+1}.\overline{z+2}}+\frac{1.2.3}{4z.\overline{z+1}.\overline{z+2}.\overline{z+3}}+\text{&c.}$$

Quæ

Quæ ponendo $A=\frac{1}{z}$, $B=\frac{A}{z+1}$, $C=\frac{2B}{z+2}$, $D=\frac{3C}{z+3}$, $E=\frac{4D}{z+4}$, + &c. evadit $A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D$ + &c. Jam ſi pro z ſubſtituatur ejus valor decimus tertius 13, habebitur Summa omnium Terminorum in Serie ſummanda poſt decimum ſecundum; quo in caſu erit $A=\frac{1}{13}$, $B=\frac{1}{14}A$, $C=\frac{2}{15}B$, $D=\frac{3}{16}C$, $E=\frac{4}{17}D$, &c. Et $A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D$ + &c. erit Summa Terminorum $\frac{1}{169}+\frac{1}{196}+\frac{1}{225}+\frac{1}{256}$ + &c. Computus autem ſic ſe habet

A = .076923077	.076923077
B = - 5494505	2747252
C = - - 732601	244200
D = - - 137363	34341
E = - - - 32321	6464
F = - - - 8978	1496
G = - - - 2835	405
H = - - - - 992	124
I = - - - - 378	42
K = - - - - 155	16
L = - - - - 67	6
M = - - - - 31	3
N = - - - - 15	1
	.079957427

Unde prodit .079957427 pro Summa Terminorum $\frac{1}{169}+\frac{1}{196}+\frac{1}{225}$ + &c. quæ adjecta aggregato duodecim initialium, ſive 1.564976638, conficit 1.644934065 pro valore totius Seriei $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}$ + &c. Convergit autem hæc Series minus celeriter ea *Brounkeri* in Exemplo priore.

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Propositio III.

Si Termini cujuſvis Seriei formentur ſcribendo numeros quoſlibet unitate differentes pro z in Quantitate

$$x^{z+n} \text{ in } \frac{a}{z} + \frac{b}{z.z+1} + \frac{c}{z.z+1.z+2} + \frac{d}{z.z+1.z+2.z+3} + \&c.$$

erit Summa æqualis x^{z+n} *in*

$$\frac{a}{1-x.z} + \frac{b-Ax}{1-x.z.z+1} + \frac{c-2Bx}{1-x.z.z+1.z+2} + \frac{d-3Cx}{1-x.z.z+1.z+2.z+3} \&c.$$

Quantitates A, B, C, D, &c. deſignant Coefficientes Terminorum præcedentes eos in quibus reperiuntur; ſcilicet eſt $A=\frac{a}{1-x}$, $B=\frac{b-Ax}{1-x}$, $C=\frac{c-2Bx}{1-x}$, &c. & ſic porro. Excipio autem caſum in quo eſt x æqualis unitati: ubi hoc accidit, Series ſummabitur per Propoſitionem ſuperiorem. Item vero demonſtrationem. Finge Summam

$$S = x^{z+n} \text{ in } \frac{A}{z} + \frac{B}{z.z+1} + \frac{C}{z.z+1.z+2} + \frac{D}{z.z+1.z+2.z+3} + \&c.$$

deinde ſcribe valores variabilium ſuccedentes S—T & $z+1$ pro antecedentibus S & z reſpective, atque habebis

$S-T = x^{z+n+1}$ in

$$\frac{A}{z+1} + \frac{B}{z+1.z+2} + \frac{C}{z+1.z+2.z+3} + \frac{D}{z+1.z+2.z+3.z+4} + \&c.$$

Hoc eſt, $S = x^{z+n}$ in

$$\frac{Ax}{z+1} + \frac{Bx}{z+1.z+2} + \frac{Cx}{z+1.z+2.z+3} + \frac{Dx}{z+1.z+2.z+3.z+4} + \&c.$$

quæ reducta ad formam ipſius S, evadit

$S - T = x^{z+n}$ in

$$\frac{Ax}{z} + \frac{Bx-Ax}{z.z+1} + \frac{Cx-2Bx}{z.z+1.z+2} + \frac{Dx-3Cx}{z.z+1.z+2.z+3} + \&c.$$

Subdu-

Subducito nunc valorem ipſius S — T a valore ipſius S, & relinquetur Terminus

$$T = x^{z+n} \text{ in } \frac{A\overline{1-x}}{z} + \frac{B\overline{1-x}+Ax}{z.\overline{z+1}} + \frac{C\overline{1-x}+2Bx}{z.\overline{z+1}.\overline{z+2}} + \frac{D\overline{1-x}+3Cx}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Hic denique valor ipſius T collatus cum illo in Propoſitione, dat $A\overline{1-x} = a$, $B\overline{1-x} + Ax = b$, $C\overline{1-x} + 2Bx = c$, & ita porro. Quæ Æquationes exhibent valores Coefficientium ut ſupra. Quare valor Summæ recte aſſignatur. Q. E. D.

Exemplum I.

Summanda ſit Series $1 + \frac{1}{3}t + \frac{1}{5}t^2 + \frac{1}{7}t^3 + \frac{1}{9}t^4 + \&c.$ Æquatio ad eandem eſt $T = t^{z-\frac{1}{2}}$ in $\frac{\frac{1}{2}}{z}$, nam ſcribendo $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, &c. ſucceſſive pro z, provenient Termini Seriei. Comparando autem hunc Terminum cum illo in Theoremate; erit $x = t$, $n = -\frac{1}{2}$, $a = \frac{1}{2}$; at b, c, d, e, & Coefficientes reliquæ ſunt nihil. Et hiſce demum valoribus ſcriptis, oritur

$$S = t^{z-\frac{1}{2}} \text{ in } \frac{\frac{1}{2}}{\overline{1-t}.z} + \frac{At}{\overline{t-1}.z.\overline{z+1}} + \frac{2Bt}{\overline{t-1}.z.\overline{z+1}.\overline{z+2}} + \frac{3Bt}{\overline{t-1}.z.\overline{z+1}.\overline{z+2}.\overline{z+3}} \&c.$$

Ponamus Exempli gratia eſſe $t = -1$; & Series ſummanda erit $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.$ atque evadet Summa

$$S = \pm 1 \text{ in } \frac{1}{4z} + \frac{A}{2z.\overline{z+1}} + \frac{2B}{2z.\overline{z+1}.\overline{z+2}} + \frac{3C}{2z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

$$\text{Vel } S = \pm 1 \text{ in } \frac{1}{4z} + \frac{A}{2z+2} + \frac{2B}{2z+4} + \frac{3C}{2z+6} + \frac{4D}{2z+8} + \&c.$$

Ubi A, B, C, D, &c. jam deſignant totos Terminos more *Newtoniano*, & non amplius Coefficientes. Et unitas cum ſigno ambiguo in quam tota Series multiplicatur, erit affirmativa ubi $z - \frac{1}{2}$ eſt numerus par, & negativa ubi eſt impar. Collige jam duodecem Terminos initiales, vel quod perinde eſt, ſex in hac Serie $\frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \&c.$ conjunctis

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conjunctis binis quibuſque in priore: eorumque Summam reperies .7646006915. Dein ſcribe pro z ejus valorem decimum tertium $12\frac{1}{2}$, & obtinebis

$$S=\frac{1}{50}+\frac{1}{27}A+\frac{2}{29}B+\frac{3}{31}C+\frac{4}{33}D+\frac{5}{35}E+\&c.$$

quæ eſt Series ſimplex & celeriter convergens: decem etenim Termini dant S = .0207974719 ut ex computo appoſito liquet: & eâdem adjectâ aggregato initialium, provenit .7853981634 pro valore Seriei ſummandæ: ad quem tamen nunquam pervenire liceret per additionem Terminorum. Atque colligendo plures initiales, valor ipſius S longe celerius approximabit. Hujus itaque Propoſitionis ope, Circumferentia circuli produci poteſt minimo labore ad figuras quamplurimas, per Seriem hancce utcunque lente convergentem; id quod olim multum deſiderabat *Leibnitius*.

.0200000000
7407407
510856
49438
5992
856
139
25
5
1
S = .0207974719

Periferia circuli obtinebitur quoque accuratiſſime per Seriem *Newtoni* ſequentem $1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+$ &c. ubi Termini quique bini ſunt alternatim negativi & affirmativi. Idem etiam efficitur per hancce $1+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}+\frac{1}{7}+\frac{1}{8}-\frac{1}{10}-\frac{1}{11}+$ &c. in qua Denominatores conſtituunt progreſſionem numerorum naturalium, dempto quoque terno. Prior æquatur quartæ & poſterior tertiæ totius circumferentiæ parti, ex ſuppoſitione quòd horum Arcuum Chordæ ſint unitates. Priuſquam tamen tractentur per hanc Propoſitionem, diſpertiendæ ſunt utræque in duas; prior in haſce

$$1-\frac{1}{5}+\frac{1}{9}-\frac{1}{13}+\frac{1}{17}-\frac{1}{21}+\&c.$$

$$\frac{1}{3}-\frac{1}{7}+\frac{1}{11}-\frac{1}{15}+\frac{1}{19}-\frac{1}{23}+\&c.$$

Poſterior in haſce $1-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\frac{1}{13}-\frac{1}{16}+$ &c.

$$\frac{1}{2}-\frac{1}{5}+\frac{1}{8}-\frac{1}{11}+\frac{1}{14}-\frac{1}{17}+\&c.$$

Dein

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Dein harum quatuor unaquæque ſeorſim conſideranda eſt, & operatio inſtituenda ut in Exemplo ſuperiore.

Exemplum II.

Si Series ſit $\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \frac{x^5}{9.10} +$ &c. Æquatio erit

$T = x^z$ in $\frac{1}{4z.\overline{z-\frac{1}{2}}}$, quæ reſoluta in Seriem evadit

$T = x^z$ in

$$\frac{1}{4z.\overline{z+1}} + \frac{3}{8z.\overline{z+1}.\overline{z+2}} + \frac{15}{16z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{105}{32z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$$

Unde comparando membra, erit $n=0$, $a=\frac{1}{4}$, $b=\frac{3}{8}$, $c=\frac{15}{16}$, $d=\frac{105}{32}$, $e=\frac{945}{64}$, &c.

Adeoque $S = x^z$ in

$$\frac{1}{4.\overline{1-x}.z} + \frac{3-8Ax}{8.\overline{1-x}.z.\overline{z+1}} + \frac{15-32Bx}{16.\overline{1-x}.z.\overline{z+1}.\overline{z+2}} + \frac{105-96Cx}{32.\overline{1-x}.z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Cujus Seriei progreſſio cuivis patet. Et ubi datur valor ipſius x in caſu quovis particulari, dabitur Summa pro libitu accurata; primo ſcilicet addendo numerum ſufficientem Terminorum initialium, eo fine ut z ſit ſatis magna ad efficiendum valorem ipſius S celeriter convergere. Et hiſce præmiſſis de Seriebus quarum Termini ſunt aſſignabiles, pergendum eſt ad eas quæ determinantur per relationem Terminorum.

Propositio IV.

Data relatione inter Summas ſucceſſivas, invenire eam quæ eſt inter Terminos.

In Æquatione definiente relationem inter Summas, ſubſtitue pro S′, S″, S‴, &c. ſuos valores proprios S—T, S—T—T′, S—T—T′—T″, &c. & ſic habebis Æquationem involventem Summam unicam S; in qua ſcribe valores variabilium conſequentes pro antecedentibus, & habebis Æquationem novam involventem Summam illam S: ope harum denique Æquationum eliminetur S, & ea quæ reſultat monſtrabit relationem Terminorum. Q. E. I.

Exemplum I.

Sit Æquatio ad Summas, $\overline{z-n}\,S=\overline{z-1}\,S'$; pro S' ſubſtitue ſuum valorem $S-T$, ac evadet Æquatio $\overline{n-1}\,S=\overline{z-1}\,T$; in qua ſcribe valores variabilium ſuccedentes pro antecedentibus, id eſt, $S-T$ pro S, T' pro T, & $z+1$ pro z; atque prodibit $\overline{n-1}\,S=\overline{n-1}\,T+zT'$; hanc de priore $\overline{n-1}\,S=\overline{z-1}\,T$ ſubducito, & manebit $\overline{z-n}\,T=zT'$: quæ eſt Æquatio ad Terminos Seriei.

Exemplum II.

Proponatur Æquatio ad Summas $S\times\overline{8zz+20z+9}+3S'\times\overline{8zz+4z-3}$; ſubſtitue $S-T$ pro S', ac invenies $S=\frac{3}{32}T\times\frac{8zz+4z-3}{zz+z}$: dein juxta methodum differentialem ſcribe $S-T$ pro S, T' pro T, & $z+1$ pro z, ac proveniet $S-T=\frac{3}{32}T\times\frac{8zz+20z+9}{zz+3z+2}$; ope harum Æquationum extermina S, & habebitur $\overline{z+2}\,T+3T'z=0$, quæ eſt Æquatio exhibens relationem Terminorum.

Atque eadem methodo exterminantur tres, pluresve Summæ ſucceſſivæ.

Propositio V.

Invenire Series quotvis ſummabiles.

Æquatio ad Summas dabit Summam Terminorum, ea vero ad Terminos dabit Seriem; prior pro libitu aſſumitur, & ex ea deducitur poſterior per Propoſitionem ſuperiorem: habentur ergo Termini, eorumque Summa. Q. E. I.

Exemplum I.

Sit Æquatio ad Summas $\overline{z-n}\,S=\overline{z-1}\,S'$, ut in Exemplo primo Propoſitionis præcedentis; invenies eam ad Terminos eſſe $\overline{z-n}\,T=zT'$. Æquatio autem ad Summas, ſubſtituendo $S-T$ pro S', dabit Summam $S=\frac{z-1}{n-1}T$. Deſignent jam A, B, C, D, &c. Terminos hujus Seriei, & in Æquatione ad eoſdem ſcribe, m, $m+1$, $m+2$, $m+3$, &c. ſucceſ-

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ſucceſſive pro z, exiſtente m numero quocunque integro vel fracto, negativo vel affirmativo; atque prodibunt relationes Terminorum, $B = \frac{m-n}{m} A$, $C = \frac{m-n+1}{m+1} B$, $D = \frac{m-n+2}{m+2} C$, $E = \frac{m-n+3}{m+3} D$, &c. Tum in Æquatione $S = \frac{z-1}{n-1} T$, ſcribe primum Terminum Seriei, id eſt, A pro T, & primum valorem ipſius z, id eſt, m pro z; atque invenies

$$S = \frac{m-1}{n-1} A = A + \frac{m-n}{m} A + \frac{m-n+1}{m+1} B + \frac{m-n+2}{m+2} C + \&c.$$

Ubi ſubſtituere licet quoſvis numeros pro m & n. Sit Exempli gratiâ $m = 5$, $n = 2$, $A = \frac{1}{12}$, & proveniet

$$\frac{1}{3} = \frac{1}{12} + \frac{3}{5} A + \frac{4}{6} B + \frac{5}{7} C + \frac{6}{8} D + \&c.$$

$$\text{id eſt, } \frac{1}{3} = \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} + \frac{1}{7.8} + \&c.$$

Ubi patet Terminos eſſe aſſignabiles; id quod ſemper eveniet quando n eſt numerus integer: tot vero erunt factores in Denominatoribus, quot ſunt unitates in n. Sic in præſente Exemplo, eſt $n = 2$, & propter id ſunt duo factores in Denominatoribus Terminorum.

Sit jam $m = 2$, $n = \frac{3}{2}$, $A = 1$, & prodibit

$$S = 2 = 1 + \frac{1}{4} A + \frac{3}{6} B + \frac{5}{8} C + \frac{7}{10} D + \frac{9}{12} E + \&c.$$

$$\text{hoc eſt, } S = 2 = 1 + \frac{1}{4} + \frac{1}{8} + \frac{5}{64} + \frac{7}{128} + \frac{21}{512} + \&c.$$

Nunc autem conſtat Terminos non eſſe aſſignabiles, quoniam n eſt numerus fractus. Notandum Seriem abrumpere & eſſe numero Terminorum finitam quotieſcunque eſt $m-n$ nihil vel integer & negativus. Atque ſi $n-1$ ſit nihil vel numerus negativus valor Seriei erit infinite magnus: ut conſtat ex valore Summæ, nempe $\frac{m-1}{n-1} A$.

EXEMPLUM

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Exemplum II.

Sit Æquatio ad Summas $S \times \overline{8zz+10z+9} + 3\,S' \times \overline{8zz+4z-3}$, ut in Exemplo noviſſimo Propoſitionis ſuperioris, ubi inventa eſt $S = \frac{3}{3^z} T \times \frac{8zz+4z-3}{zz+z}$, & relatio Terminorum erat $\overline{z+2}\,T + 3zT' = 0$. Et ſi uſurpetur unitas pro Termino primo, & primus valor ipſius z ponatur etiam unitas; obtinebitur

$$S = \frac{27}{64} = 1 - \frac{3}{3} + \frac{6}{9} - \frac{10}{27} + \frac{15}{81} - \frac{21}{243} + \&c.$$

Ubi Denominatores ſunt dignitates ternarii, Numeratores vero numeri triangulares. In hiſce Exemplis non expatiatus ſum in deducendis Seriebus ex Æquationibus quæ definiunt relationes Terminorum, quoniam hoc ex Introductione hactenus notum ſuppono.

Scholion.

Summatio Serierum in Methodo differentiali, reſpondet Quadraturæ Curvarum in Methodo Fluxionum; & eapropter in utraque oriri ſolent difficultates conſimiles, quæ hic ſunt enodandæ. Diximus Series hinc inde continuari poſſe in infinitum: exempli gratia Series $1 + x + x^2 + x^3 + \&c.$ retrorſum continuata eſt $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \&c.$ Et hæ duæ ſimul junctæ conſtituunt unicam ex utraque parte excurrentem in infinitum, ſcilicet

$$\&c. + \frac{1}{x^4} + \frac{1}{x^3} + \frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2 + x^3 + x^4 + \&c.$$

Nam hi Termini ſunt in progreſſione Geometrica continua, antecedentes ad conſequentes ut unitas ad x. In Summatione hujus Seriei inveniemus $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \&c.$ quod quidem verum eſt ubi x eſt minor unitate; attamen ſi ſit major unitate, hæc Series erit infinite magna, atque Summa $\frac{1}{1-x}$, non amplius erit Summa horum Terminorum; ſed mutato ſigno, æquabitur Seriei ad alteras partes excurrenti; hoc eſt, erit $\frac{-1}{1-x}$, vel $\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \&c.$ Quod ſi

ſi x ſit unitas, Summa erit $\frac{1}{1-1}$, & inde utraque pars Seriei erit infinite magna, utpote æqualis unitati infinities ſumptæ.

Ad eundem modum ſi Æquatio ad Seriem ſit $\overline{z-n}\,T = zT'$, & Series continuetur ex utraque parte in infinitum; una pars converget & altera diverget præterquam ubi eſt $n=1$; & Summa $\frac{z-1}{n-1}\,T$ ſemper æquabitur parti Seriei convergenti.

Haud ſecus in Quadraturis, ſi z^{-n} ſit Ordinata Curvæ Hyperbolicæ, fluens $\frac{1}{1-n}z^{1-n}$ exprimet partem Areæ ad has aut illas Ordinatæ partes jacentem prout n eſt minor aut major unitate: ubi tamen n eſt unitas, Area ex utraque parte Ordinatæ erit infinite magna, ut in Hyperbola *Apollonii*.

Series quidem, etiamſi quantitates quæ per eas quæruntur ſint finitæ magnitudinis, ſæpiſſime ſolent divergendo evadere infinite magnæ, in illis autem caſibus continuatæ in alteras partes nonnunquam convergunt & æquantur Radicibus quæſitis, vel ab iiſdem differunt quantitate determinata. Nonnunquam etiam utrinque continuatæ divergunt: ſæpius etiam ad utraſque partes nequeunt excurrere in infinitum propter Terminos impoſſibiles vel infinite parvos.

Inſuper ſicut Areæ Curvarum nunc augendæ nunc minuendæ ſunt datis quantitatibus, ut evadant veræ; ita etiam Summæ per hanc Propoſitionem inventæ nonnunquam differunt a veris, quo in caſu corrigendæ ſunt additione vel ablatione quantitatis datæ. Scilicet ubi Æquatio ad Summas talis eſt qualis efficit earum ultimam eſſe quantitatem magnitudinis finitæ vel infinite magnæ, ſemper opus eſt correctione: monſtrabo igitur in ſequente Propoſitione, qua ratione aſſumenda ſit Æquatio quæ ſemper efficiet ultimam eſſe nihil; eoque pacto ſumma inventa erit vera, nec augenda neque minuenda, ut hactenus monſtratum eſt.

Propositio VI.

Si Æquatio ad Summas ſit $S \times \overline{z^l + az^{l-1} + bz^{l-2} + \&c.} = m\,S' \times \overline{z^l + cz^{l-1} + dz^{l-2} + \&c.}$ *ultima Summarum erit finitæ magnitudinis in eo caſu ſolo, ubi eſt* $m=1$, *& ſimul* $a=c$.

Ad hanc Propoſitionem demonſtrandam, ſciendum eſt Summam S inveſtigari poſſe ex Æquatione definiente relationem inter eam ejuſque

L valorem

valorem ſucceſſivum S′, eodem fere modo ac quantitas Fluxionalis ex ſua Æquatione. Illum in finem aſſumenda eſt pro S Series hujus formæ,

$$\frac{z^n}{p^z} \text{ in } A + \frac{B}{z} + \frac{C}{z^2} + \frac{D}{z^3} + \&c.$$

ubi n, p, A, B, C, D, &c. ſunt quantitates invariabiles. In caſu autem præſente ubi Summa quæſita eſt omnium ultima, adeoque ad diſtantiam infinitam remota; erit z etiam infinite magna, utpote quæ vel æqualis eſt illi diſtantiæ vel ab eadem differt quantitate finita: hac de cauſa Termini Seriei poſteriores ſunt infinite minores prioribus. Igitur ad abbreviandum computum, rejicio omnes poſt primum, utpote quæ in hac demonſtratione ſunt inutiles: ſic habeo $S = \frac{Az^n}{p^z}$; in qua ſcribendo S′ pro S, & $\overline{z+1}$ pro z; obtineo $S' = \frac{A\overline{z+1}^n}{p^{z+1}}$. Hoſce valores ſubſtitue pro S & S′ in Æquatione ad Summas, vel quod eodem redit, in hac $S \times \overline{z+a} = mS \times \overline{z+c}$, membris reliquis ob rationes ſupra expoſitas neglectis; & reſultabit $\frac{Az^n}{p^z} \times \overline{z+a} = \frac{mA\overline{z+1}^n}{p^{z+1}} \times \overline{z+c}$. Vel ducendo in p^{z+1}, & dividendo per A, $pz^n \times \overline{z+a} = m \times \overline{z+1}^n \times \overline{z+c}$. Cæterum per Theorema *Newtoni* pro evolvendo Binomio, eſt $\overline{z+1}^n = z^n + nz^{n-1}$, idque accurate propter z infinite magnam. Hunc valorem ſubſtitue, & evadet Æquatio $pz^n \times \overline{z+a} = m \times \overline{z^n + nz^{n-1}} \times \overline{z+c}$, quæ diviſa per z^{n-1}, fit $pz \times \overline{z+a} = m \times \overline{z+n} \times \overline{z+c}$, vel $pzz + paz = mzz + \overline{n+c} \times mz + mnc$; in qua conferendo membra homologa, erit $p = m$ & $pa = \overline{n+c} \times m$, unde $n = a - c$: & proinde ultima Summa S, quæ antea erat $\frac{Az^n}{p^z}$, jam evadit $\frac{Az^{a-c}}{m^z}$, ubi notandum eſt Coefficientem A non determinari. Fingatur nunc $m = 1$, & ſimul $a = c$; & Summa fiet $\frac{Az^0}{1^z}$: eſt vero $z^0 = 1$, & $1^z = 1$, etiamſi z ſit infinite magna; atque adeo ultima Summarum eſt finita, quippe æqualis quantitati A ubi eſt $m = 1$, & ſimul $a = c$: alius autem non datur caſus in quo $\frac{Az^{a-c}}{m^z}$ eſt quantitas finita ubi z eſt infinite magna. Conſtat igitur Propoſitio.

Corol-

Corollarium.

Si m ſit minor unitate, ultima Summa erit infinite magna; & infinite parva ubi m eſt major unitate. Et ſi m ſit unitas, Summa illa erit infinite magna vel parva prout a eſt major aut minor c. Igitur in Æquatione ad Summas ſi m ſit major unitate, vel unitati æqualis & ſimul a minor c; ultima Summa ſemper erit nihil & nulla Correctione opus erit.

Exemplum.

Sit Summa prima $A = 1$, ſecunda $B = \frac{9}{8} A$, tertia $C = \frac{25}{24} B$, quarta $D = \frac{49}{48} C$, quinta $E = \frac{81}{80} D$, &c. Atque Æquatio ad eaſdem erit $S \times \overline{zz+z+\frac{1}{4}} = S' \times \overline{zz+z}$; quæ collata cum Æquatione generali, dat $m = 1$, $a = 1$, $c = 1$; & inde $a - c = 0$; unde per hanc Propoſitionem ultima Summarum, id eſt factum ſub omnibus numeris

$$1 \times \frac{9}{8} \times \frac{25}{24} \times \frac{49}{48} \times \frac{81}{80} \times \frac{121}{210} \times \text{ \&c. in infinitum, eſt quantitas finita.}$$

In Æquatione ad Summas ſubſtitue $S - T$ pro S, & invenies $S = -4T \times \overline{zz+z}$, quæ quoniam prodit negativa reſpectu ipſius T, exhibet Summam Terminorum non a dato Termino uſque in infinitum, ſed a dato Termino uſque ad principium Seriei.

Quod ut clarius pateat, proponantur duæ Æquationes $Szz = S' \times \overline{zz-1}$, & $Sz = S'\overline{z+1}$: utravis harum dabit eandem Æquationem ad Terminos, ſcilicet $Tz = T'\overline{z+2}$. Ex priore tamen deducitur $S = -T \times \overline{zz+1}$, & ex poſteriore $S = T\overline{z+1}$. In primo caſu S eſt Summa Terminorum ab initio uſque ad T; & in ſecundo S eſt Summa ipſius T & omnium ſequentium in infinitum. In Æquatione ad Terminos ſcribantur numeri 1, 2, 3, 4, &c. ſucceſſive pro z, & uſurpetur $\frac{1}{2}$ pro primo Termino; atque prodibit Series

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} + \frac{1}{7.8} + \text{\&c.}$$

Ubi ſi quæratur aggregatum primorum quatuor Terminorum, ſcribe 5 pro z in $T \times \overline{zz-1}$ valore Summæ S priore, & Terminum quintum $\frac{1}{5.6}$ pro T, atque obtinebis $\frac{25-1}{5.6} = \frac{4}{5} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}$,

quatuor

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quatuor utique Terminis initialibus. At ſi vis Summam omnium omnino Terminorum, præter illos quatuor, ſcribe 5 pro z, & $\frac{1}{5.6}$ pro T, in $T_{\overline{z+1}}$ valore ipſius S poſteriore; ac habebis $\frac{6}{5.6} = \frac{1}{5} = \frac{1}{5.6} + \frac{1}{6.7} + \frac{1}{7.8} + \frac{1}{8.9} +$ &c. Et hi duo valores ipſius S ſimul additi, hoc eſt, Summa numerorum $\frac{4}{5}$ & $\frac{1}{5}$ conficit $\frac{4+1}{5} = 1$ pro valore omnium omnino Terminorum ab initio uſque ad infinitum.

Propositio VII.

Si Æquatio ad Seriem ſit $\overline{z-n}\,T + \overline{m-1}.z\,T'$, *erit*
$$S = \frac{m-1}{m}T + \frac{n}{z}\times\frac{A}{m} + \frac{n+1}{z+1}\times\frac{B}{m} + \frac{n+2}{z+2}\times\frac{C}{m} + \frac{n+3}{z+3}\times\frac{D}{m} + \frac{n+4}{z+4}\times\frac{E}{m} + \&c.$$

Finge Summam S æqualem eſſe Termino T ducto in quantitatem y, hoc eſt $S = Ty$; dein ſcribe valores indeterminatarum poſteriores $S-T$, T', y' pro prioribus S, T, & y reſpective, atque habebitur $S - T = T'y'$; quæ ablata de priore $S = Ty$, relinquit $T = Ty - T'y'$, unde eſt $T' = T\times\frac{y-1}{y'}$. Sed per Æquationem ad Seriem ſcilicet $\overline{z-n}\,T + \overline{m-1}.z\,T'$, eſt $T' = -\frac{\overline{z-n}\,T}{z.\overline{m-1}}$. Adeoque æquando ſibi invicem duos ipſius T′ valores, erit $T\times\frac{y-1}{y'} = -\frac{\overline{z-n}\,T}{z.\overline{m-1}}$: quæ diviſa per T, & ducta in $\overline{m-1}.y'$, evadit $\overline{m-1}.y - m + 1 = -y' + \frac{n}{z}y'$, vel $\overline{m-1}.y + y' - \frac{n}{z}y' - m + 1 = 0$: quæ eſt Æquatio Differentialis ex cujus reſolutione dabitur Radix y. Illum in finem aſſume
$$y = a + \frac{b}{z} + \frac{c}{z.\overline{z+1}} + \frac{d}{z.\overline{z+1}.\overline{z+2}} + \frac{e}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$
tum ſcribe pro y & z, valores earum ſuccedaneos y' & $z+1$ reſpective, atque habebis

$y' = a$

$y' = a +$

$$\frac{b}{\overline{z+1}} + \frac{c}{\overline{z+1}.\overline{z+2}} + \frac{d}{\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{e}{\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$$

hoc eſt, $y' = a + \frac{b}{z} + \frac{c-b}{z.\overline{z+1}} + \frac{d-2c}{z.\overline{z+1}.\overline{z+2}} + \frac{e-3d}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$

Atque prior valor ipſius y' multiplicatus per $\frac{n}{z}$, evadit

$$\frac{n}{z}y' = \frac{na}{z} + \frac{nb}{z.\overline{z+1}} + \frac{nc}{z.\overline{z+1}.\overline{z+2}} + \frac{nd}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Hoſce valores ad eandem formam jam reductos ſubſtitue in Æquatione, & reſultabit

$$ma - m + 1 + \frac{mb - na}{z} + \frac{mc - \overline{n+1}.b}{z.\overline{z+1}} + \frac{md - \overline{n+2}.c}{z.\overline{z+1}.\overline{z+2}} + \&c. = 0.$$

Ubi ponendo membra homologa æqualia nihilo, habebitur $a = \frac{m-1}{m}$, $b = \frac{n}{m}a$, $c = \frac{n+1}{m}b$, $d = \frac{n+2}{m}c$, $e = \frac{n+3}{m}d$, &c. Et hiſce datis, dabitur valor Radicis y, quæ denique ducta in T exhibebit pro S, Seriem in Propoſitione exhibitam. Q. E. D.

Corollarium. Si n ſit numerus integer & negativus, vel nihil; valor ipſius S abrumpet, exiſtente Serie ſummabili. Et ubi m eſt negativus, Series erit infinite magna. Hic autem excipio caſum in quo eſt $m = 0$; tum enim Series ſummabitur per Exemplum primum Propoſitionis quintæ.

Exemplum I.

Proponatur Series ſummanda $\frac{1}{2} + \frac{1}{3}A + \frac{2}{5}B + \frac{3}{7}C + \frac{4}{9}D + \frac{5}{11}E + \&c.$ Æquatio eam definiens eſt $\overline{z - \frac{1}{2}}\,T - 2zT' = 0$, ubi valores ſucceſſivi ipſius z ſunt $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, &c. & illa collata cum Æquatione generali dat $n = \frac{1}{2}$, $m - 1 = -2$, ſive $m = -1$; quibus ſubſtitutis, oritur

M S =

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$$S = 2\,T - \frac{A}{2z} - \frac{3B}{2z+2} - \frac{5C}{2z+4} - \frac{7D}{2z+6} - \frac{9E}{2z+8} - \&c.$$

In qua ſi ſubſtituatur quilibet Terminus pro T, & pro z valor ſuus correſpondens, erit S Summa ipſius T & Terminorum omnium ſequentium uſque in infinitum. Colligo itaque duodecem Terminos initiales; eorumque aggregatum emergit .78533961813. Dein ut obtineam Summam reliquorum, ſcribo Terminum decimum tertium, id eſt, .00003029411 pro T, & pro z valorem ſuum debitum $13\frac{1}{2}$; atque habeo

$$S = .00006058822 - \frac{1}{27}A - \frac{3}{29}B - \frac{5}{31}C - \frac{7}{33}D - \frac{9}{35}E - \&c.$$

Ubi Termini prodeunt alternatim negativi & affirmativi, hoſce diſpono in una columna & illos in altera ut vides

.00006058822	.00000224401
23214	3744
794	204
61	20
7	3
1	1
+ .00006082899	− .00000228373

Tum Summam negativorum .00000228373 auferens ab ea affirmativorum .00006082899, habeo S = .00005854526, quæ adjecta Summæ initialium exhibet .78539816339 pro valore Seriei propoſitæ, id eſt, pro Area Circuli cujus Diamiter eſt unitas.

Exemplum II.

Quæratur valor Seriei $1 - \frac{1}{2}A - \frac{3}{4}B - \frac{5}{6}C - \frac{7}{8}D - \frac{9}{10}E - \&c.$

Æquatio definiens relationem Terminorum eſt $\overline{z - \frac{1}{2}}\,T + zT' = 0$, in qua valores Abſciſſæ z ſunt 1, 2, 3, 4, &c. comparando autem hanc Æquationem cum illa in Theoremate, erit $n = \frac{1}{2}$, $m - 1 = 1$, & $m = 2$: atque exinde

$$S = \frac{1}{2}T + \frac{A}{4z} + \frac{3B}{4z+4} + \frac{5C}{4z+8} + \frac{7D}{4z+12} + \frac{9E}{4z+16} + \&c.$$

Collige

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Collige nunc decem Terminos initiales Seriei tranſmutandæ, eorumque Summam reperies fore .6168670654. Deinde ut habeatur Summa reliquorum; in valore ipſius S ſcribe Terminum decimum primum, id eſt, .1761970520 pro T & 11 pro z; atque prodibit

$$S = .0880985260 + \frac{1}{44}A + \frac{3}{48}B + \frac{5}{52}C + \frac{7}{56}D + \frac{9}{60}E + \&c.$$

Et ineundo computum ut in margine, invenietur S = .0902397156, quæ addita aggregato initialium prius invento efficit .7071067810 pro valore Seriei, id eſt pro $\sqrt{\frac{1}{2}}$. Etenim $\sqrt{\frac{1}{2}}$ ſic ſcripta $\overline{1+1}|^{-\frac{1}{2}}$, & evoluta per Theorema *Newtoni*, evadit Series de qua nunc egimus.

.0880985260
20022392
1251400
120327
15041
2250
388
74
15
3
S = .0902397156

SCHOLION.

Omnis Series cujus Termini ſunt per vices negativi & affirmativi, ſi tranſmutetur per hanc Propoſitionem, migrabit in aliam celerius convergentem cujus Termini ſunt ejuſdem ſigni. Et è contra omnis Series cujus Termini ſunt ejuſdem ſigni, abibit in aliam cujus Termini ſunt alternatim negativi & affirmativi; quæ tamen non converget celerius priore præterquam ubi Tranſmutatio inchoatur a Terminis ab initio ſatis remotis. Et ſi Series tranſmutetur, & rurſus tranſmutetur ea quæ ex prima Tranſmutatione prodiit, emerget ea prius propoſita. Exempli gratia ſi Series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.$ tranſmutetur, prodibit $\frac{1}{2} + \frac{1}{3}A + \frac{2}{5}B + \frac{3}{7}C + \frac{4}{9}D + \&c.$ dein ſi noviſſima tranſmutetur emerget prima $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c.$ hoc eſt, transfert Seriem de Tangente ad Sinum, vel de Sinu ad Tangentem. Sed in hiſce caſibus debet operatio ordiri ad primum Terminum Seriei, nempe ubi agitur de Tranſmutatione totius.

PROPO-

Propositio VIII.

Si Æquatio ad Seriem ſit $T' = \frac{z-m}{z} \times \frac{z-n}{z-n+1} T$, & *ponatur* $A = \overline{z-n}\; T$, $B = \frac{n}{z} A$, $C = \frac{n+1}{z+1} B$, $D = \frac{n+2}{z+2} C$, $E = \frac{n+3}{z+3} D$, &c. *Erit* $S = \frac{A}{m} + \frac{B}{m+1} + \frac{C}{m+2} + \frac{D}{m+3}$ &c.

Etenim ſuppone $S = T \times \overline{z-n} \times y$, & per methodum Differentialem erit $S - T = T' \times \overline{z-n+1} \times y'$, harum Æquationum Differentia dabit $T = T \times \overline{z-n} \times y - T' \times \overline{z-n+1} \times y'$. Pro T' ſcribe ſuum valorem $\frac{z-m}{z} \times \frac{z-n}{z-n+1} T$, atque habebitur $T = T \times \overline{z-n} \times y - T' \times \frac{z-m}{z} \times \overline{z-n} \times y'$ quæ diviſa per $T \times \overline{z-n}$, evadit $\frac{1}{z-n} = y - \frac{z-m}{z} y'$, vel $y - y' + \frac{m}{z} y' - \frac{1}{z-n} = 0$. Ex hac vero Æquatione eruetur Radix y ad modum ſequentem. Finge

$$y = \frac{a}{m} + \frac{b}{\overline{m+1}.z} + \frac{c}{\overline{m+2}.z.\overline{z+1}} + \frac{d}{\overline{m+3}.z.\overline{z+1}.\overline{z+2}} + \text{\&c.}$$

Erit $y' =$

$$\frac{a}{m} + \frac{b}{\overline{m+1}.\overline{z+1}} + \frac{c}{\overline{m+2}.\overline{z+1}.\overline{z+2}} + \frac{d}{\overline{m+3}.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \text{\&c.}$$

Et $y - y' =$

$$\frac{b}{\overline{m+1}.z.\overline{z+1}} + \frac{2c}{\overline{m+2}.z.\overline{z+1}.\overline{z+2}} + \frac{3d}{\overline{m+3}.z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \text{\&c.}$$

Atque $\frac{m}{z} y' =$

$$\frac{am}{mz} + \frac{bm}{\overline{m+1}.z.\overline{z+1}} + \frac{cm}{\overline{m+2}.z.\overline{z+1}.\overline{z+2}} + \frac{dm}{\overline{m+3}.z.\overline{z+1}.\overline{z+2}.\overline{z+3}} \text{ \&c.}$$

Adeoque

Adeoque $y - y' + \frac{m}{z} y' =$

$$\frac{a}{z} + \frac{c}{z.\overline{z+1}} + \frac{b}{z.\overline{z+1}.\overline{z+2}} + \frac{d}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Eſt vero $\frac{1}{z-n} =$

$$\frac{1}{z} + \frac{n}{z.\overline{z+1}} + \frac{n.\overline{n+1}}{z.\overline{z+1}.\overline{z+2}} + \frac{n.\overline{n+1}.\overline{n+2}}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Igitur $y - y' + \frac{m}{z} y - \frac{1}{z-n} =$

$$\frac{a-1}{z} + \frac{b-n}{z.\overline{z+1}} + \frac{c - n.\overline{n+1}}{z.\overline{z+1}.\overline{z+2}} + \frac{d - n.\overline{n+1}.\overline{n+2}}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c. = 0.$$

Pone jam Numeratores nihil, ut evaneſcant Termini; & pro determinatione Coefficientium habebis Æquationes ſequentes $a = 1$, $b = n$, $c = n.\overline{n+1}$, $d = n.\overline{n+1}.\overline{n+2}$, &c. ſubſtitue hoſce valores in Serie aſſumpta pro y, vice ipſarum a, b, c, d, &c. & valor ipſius y, exinde ortus ductus in $\overline{z-n}\,T$, exhibebit pro S, Seriem in Theoremate poſitam. Q. E. D.

Corallarium. Si n ſit integer & negativus, vel nihil; Series ſummabitur accurate per hoc Theorema. Et ſi m ſit nihil vel negativus, Series erit infinite magna. Inſervit autem hæc Propoſitio & ſuperior Quadraturæ Curvarum binomialium. Hæc uſui venit ubi in Ordinata $x^{\theta} \times \overline{e + f x^{\eta}}^{\lambda}$, eſt $e + f x^{\eta} = 0$; ſuperior vero adhibenda eſt ubi contrarium accidit.

Exemplum I.

Quæratur valor Seriei $1 + \frac{1.1}{2.3} A + \frac{3.3}{4.5} B + \frac{5.5}{6.7} C + \frac{7.7}{8.9} D + \frac{9.9}{10.11} E + \&c.$ Æquatio eam definiens eſt $T' = \frac{z - \frac{1}{2}}{z} \times \frac{z - \frac{1}{2}}{z + \frac{1}{2}} T$, uti patebit ſcribendo valores 1, 2, 3, 4, &c. ſucceſſive pro z. Æquatio autem in Theoremate comparata cum hac, dat $m = \frac{1}{2}$, $n = \frac{1}{2}$; unde $A = \overline{z - \frac{1}{2}}\,T$, $B =$

$\frac{A}{2z}$,

$\frac{A}{2z}$, $C = \frac{3B}{2z+2}$, $D = \frac{5C}{2z+4}$, $E = \frac{7D}{2z+6}$, &c. atque $S = \frac{2}{1}A + \frac{2}{3}B + \frac{2}{5}C + \frac{2}{7}D + \frac{2}{9}E +$ &c. Ineundo computum reperio aggregatum duodecem Terminorum eſſe 1.407397508. Dein ſubſtituo Terminum decimum tertium, id eſt, $\frac{1}{25} \times .161180258$ pro T, & pro z valorem ſuum correſpondentem 13, ſic prodit

$A = \frac{1}{2} \times .161180258$,	.166180258
$B = \frac{1}{2} \times$ - 6199241,	2066414
$C = \frac{1}{2} \times$ - - 664204,	132841
$D = \frac{1}{2} \times$ - - 110701,	15814
$E = \frac{1}{2} \times$ - - - 24261,	2691
$F = \frac{1}{2} \times$ - - - 6410,	583
$G = \frac{1}{2} \times$ - - - 1959,	151
$H = \frac{1}{2} \times$ - - - 670,	45
$I = \frac{1}{2} \times$ - - - - 250,	15
$K = \frac{1}{2} \times$ - - - - 102,	5
$L = \frac{1}{2} \times$ - - - - 44,	2
$M = \frac{1}{2} \times$ - - - - 20,	1
	$S = .163398820$

Ex hoc computo habeo $S = .163398820$, valorem utique Terminorum omnium poſt decimum ſecundum; quem itaque adjiciens Summæ initialium, 1.407397508, habeo 1.570796328 pro valore Seriei ſummandæ, id eſt, pro longitudine Arcus ſemicircularis cujus Diameter eſt unitas.

Termini initiales, quando eorum Summa eruitur, facillime reducuntur in fractiones decimales per regulam ſequentem, pone $a = 1$, $b = a - \frac{1}{2}a$, $c = b - \frac{1}{4}b$, $d = c - \frac{1}{6}c$, $e = d - \frac{1}{8}d$, &c. eruntque Termini a, $\frac{1}{3}b$, $\frac{1}{5}c$, $\frac{1}{7}d$, $\frac{1}{9}e$, &c.

Exemplum II.

Proponatur Series *Brounkeri* $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} +$ &c. Æquatio

quatio ad eandem eſt $T' = \frac{z-1}{z} \times \frac{z-\frac{1}{2}}{z+\frac{1}{2}} T$, vel etiam $T' = \frac{z-1}{z} \times \frac{z-\frac{3}{2}}{z-\frac{1}{2}} T$, ſcilicet ſumendo initium Abſciſſæ z a diverſis punctis. In priore valores ipſius z ſunt $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, $\frac{9}{2}$, &c. $m = 1$, $n = \frac{1}{2}$, $A = \overline{z - \frac{1}{2}}\, T$, $B = \frac{A}{2z}$, $C = \frac{3B}{2z+2}$, $D = \frac{5C}{2z+4}$, &c. In poſteriore valores ipſius z ſunt 2, 3, 4, 5, &c. $m = 1$, $n = \frac{3}{2}$, $A = \overline{z - \frac{3}{2}}\, T$, $B = \frac{3A}{2z}$, $C = \frac{5B}{2z+2}$, $D = \frac{7C}{2z+4}$, &c. & in utroque erit $S = A + \frac{1}{2} B + \frac{1}{3} C + \frac{1}{4} D +$ &c. Dabitur ergo S ad plures figuras ſumendo pro T quemvis è Terminis ſat longe diſtantibus a principio. Et ſimiliter in aliis caſibus inſtituere licet computum, idque duobus modis diverſis ubi Termini ſunt aſſignabiles. Hæc autem Series hunc in modum ſcripta $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} -$ &c. facilius tractabitur per Propoſitionem tertiam aut ſeptimam.

Scholion.

Hactenus egimus de Summatione Serierum quæ prodeunt in Quadratura Curvarum binomialium, & ſimilibus. Cæterum procedere licet eodem modo in caſibus difficilioribus: nam Summa Seriei determinatur & erui poteſt ex data relatione Terminorum; idque reſolvendo Æquationem Differentialem ut in noviſſimis duabus Propoſitionibus. Problema autem eſſet laborioſum quærere Summas independenter a Terminis, quando Termini non ſunt aſſignabiles; ideoque quæſivi quantitatem quæ ducta in Terminum T exhibet Summam S. Sic etiam Areæ Curvarum facilius eruuntur mediantibus Ordinatis, nam Series hac ratione prodeuntes ſunt ſimpliciſſimæ. Et hiſce præmiſſis pergo ad methodum reſolvendi Radices Æquationum Differentialium in Series magis quidem compoſitas, at ſimul longe magis convergentes quam ſuperiores; quas ea tantùm de cauſa exhibui, quod eſſent ſimplices & uſibus vulgaribus ſufficientes.

Hucuſque Summam quamſibet deſignavimus per S, ejuſque Terminos per T, T', T'', &c. At in ſequentibus etiam deſignabimus Series vel Summas, per S2, S3, S4, &c. earumque Terminos per T2, T'2, T''2,

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T″2, &c. T3, T′3, T″3, &c. & ſic porro: adeo ut ſit

$$S2 = T2 + T'2 + T''2 + \&c.$$
$$S3 = T3 + T'3 + T''3 + \&c.$$
$$S4 = T4 + T'4 + T''4 + \&c.$$
&c.

Et ſicut in Serie S, Summæ ſucceſſivæ deſignantur per S′, S″, &c. ita in Serie S2, denotabuntur per S′2, S″2, &c. atque in Serie S3 per S′3, S″3, &c. Et ſic in cæteris. Hanc autem notationem introducere coactus ſum, propterea quod Summæ & Termini diverſarum Serierum ſimul conſiderandæ veniunt in eadem Æquatione.

Propositio IX.

Data relatione inter duas Summas in diverſis Seriebus, & Æquatione ad Terminos in alterutra; invenire Æquationem ad Terminos in altera.

Problema ſolvitur progrediendo ex relatione variabilium præſente ad ſuccedentem, ut exinde eliminentur Summæ; id quod Exemplis patebit.

Exemplum I.

Sint S & S2 duæ Summæ in diverſis Seriebus, earumque relatio $S = \frac{\overline{m+1}.z-n}{m.\overline{m+1}} T + S2$; ſit vero $T' = \frac{z-m}{z} \times \frac{z-n}{z-n+1} T$, relatio Terminorum Seriei S: atque ex hiſce datis opporteat eruere Æquationem ad Terminos Seriei S2. In Æquatione $S = \frac{\overline{m+1}.z-n}{m.\overline{m+1}} T + S2$, exhibente relationem Summarum, ſubſtitue valores variabilium conſequentes pro antecedentibus, hoc eſt, S′ ſive S − T pto S; S′2, ſive S2 − T2 pro S2; T′ pro T, & $z+1$ pro z; atque habebis $S - T = \frac{\overline{m+1}.\overline{z+1}-n}{m.\overline{m+1}} T' + S2 - T2$, quæ ablata de priore relinquit Æquationem liberatam a Summis, ſcilicet $T = \frac{\overline{m+1}.z-n}{m.\overline{m+1}} T - \frac{\overline{m+1}.\overline{z+1}-n}{m.\overline{m+1}} T' + T2$; unde eſt $T2 = \frac{\overline{m+1}.\overline{z+1}-n}{m.\overline{m+1}} T' - \frac{\overline{m+1}.\overline{z-m}-n}{m.\overline{m+1}} T$: in qua ſubſti-

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ſubſtitue pro T′ valorem proprium $\frac{z-m}{z}\times\frac{z-n}{z-n+1}T$, atque obtinebis $T2=\frac{n.\overline{m-n+1}}{\overline{m+1}.z.\overline{z-n+1}}T$. In hoc inſuper valore ſcribe valores indeterminatarum ſuccedentes T′2, T′, & $z+1$ pro præſentibus T2, T, & z, & orietur $T'2=\frac{n.\overline{m-n+1}}{\overline{m+1}.\overline{z+1}.\overline{z-n+2}}T'$; ubi pro T′ rurſus ſubſtituendo ſuum valorem, emergit $T'2=\frac{n.\overline{m-n+1}}{\overline{m+1}.\overline{z+1}.\overline{z-n+2}}T\times\frac{z-m}{z}\times\frac{z-n}{z-n+1}$, ex qua prodit $T=\frac{\overline{m+1}.z.\overline{z+1}.\overline{z-n+1}.\overline{z-n+2}}{n.\overline{m-n+1}.\overline{z-m}.\overline{z-n}}T'2$. Æquatio autem prius inventa $T2=\frac{n.\overline{m-n+1}}{\overline{m+1}.z.\overline{z-n+1}}T$ dat $T=\frac{\overline{m+1}.z.\overline{z-n+1}}{n.\overline{m-n+1}}T2$. Æquentur jam ſibi invicem duo valores Termini T, & incides in Æquationem $T'2=\frac{z-m}{z+1}\times\frac{z-n}{z-n+2}T2$, quæ exprimit relationem Terminorum Seriei S2. Q. E. I.

Exemplum II.

Sit jam relatio inter Summas $S=\frac{3}{4}\times\frac{4z+3}{4z+2}T+S2$, & $zT+3T'\times\overline{z+1}$ Æquatio ad Seriem S. In Æquatione ad Summas ſcribe valores variabilium ſucceſſivos, & habebitur $S-T=\frac{3}{4}\times\frac{4z+7}{4z+6}T'+S2-T2$: quam de priore auferens habebis $T=\frac{3}{4}\times\frac{4z+3}{4z+2}T-\frac{3}{4}\times\frac{4z+7}{4z+6}T'+T2$; ex qua comperies $T2=\frac{1}{4}\times\frac{4z-1}{4z+2}T+\frac{3}{4}\times\frac{4z+7}{4z+6}T'$. Cæterum per Æquationem ad Seriem S, eſt $3T'=-\frac{z}{z+1}T$, quo ſcripto, prodit $T2=+\frac{1}{4}\times\frac{4z-1}{4z+2}T+\frac{1}{4}\times\frac{4z+7}{4z+6}\times\frac{-z}{z+1}T$, vel $T2=-\frac{3}{4}T\times\frac{1}{2z+1}\times\frac{1}{2z+2}\times\frac{1}{2z+3}$. Atque rurſus

ſus progrediendo ad valores indeterminatarum ſuccedentes, habebitur $T'2 = -\frac{3}{4} T' \times \frac{1}{2z+3} \times \frac{1}{2z+4} \times \frac{1}{2z+5}$, vel ſubſtituendo $-\frac{z}{z+1} T$ pro $3T'$, $T'2 = \frac{1}{2} \times \frac{1}{2z+2} \times \frac{1}{2z+3} \times \frac{1}{2z+4} \times \frac{1}{2z+5} T$: unde $T = 2 . \overline{2z+2} . \overline{2z+3} . \overline{2z+4} . \overline{2z+5} . T'2$; atque per valorem ipſius $T2$ eſt idem $T = -\frac{4}{3} \times \overline{2z+1} \times \overline{2z+2} \times \overline{2z+3}\, T2$; quibus duobus valoribus inter ſe æquatis, obtinebitur $\overline{zz+\frac{1}{2}z} \times T2 + \overline{zz+\frac{9}{2}z+5} \times 3T'2 = 0$. Æquatio utique ad Terminos Seriei S. Et in aliis caſibus ſimiliter procedendum eſt.

SCHOLION.

Per hanc Propoſitionem Series infinitæ inter ſe comparari poſſunt. Nam Æquatio exhibens relationem inter Summas S & $S2$, dabit unam ex data altera: & Termini ipſarum S & $S2$, dabuntur ex ſuis Æquationibus, quarum una aſſumitur, altera vero ex ea aſſumpta eruitur: ut in Exemplis hujus Propoſitionis. Ut ſi relatio Summarum ſit $S = \frac{2z-1}{2} T + S2$, & Æquatio ad Seriem S, $T' = \frac{zz-2z+1}{zz} T$; invenies $T2 = \frac{T}{2zz}$, & $T'2 = \frac{zz-2z+1}{zz+2z+1} T2$ Æquationem ad Seriem $S2$. Sint jam 2, 3, 4, 5, &c. valores ſuccedentes indeterminatæ z; & uſurpando unitatem pro T invenies

$$S = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \&c.$$

$$S2 = \frac{1}{8} \text{ in } 1 + \frac{1}{9} + \frac{1}{36} + \frac{1}{100} + \frac{1}{225} + \&c.$$

In priore Denominatores ſunt quadrata numerorum naturalium, & in poſteriore quadrata numerorum Triangularium. Relatio autem inter Series $S = \frac{2z-1}{2} T + S2$, ſcribendo 2 pro z, & unitatem pro T, dabit $S2 = S - \frac{3}{2}$, & ducendo in 8, erit $8S - 12 = 8S2 = 1 + \frac{1}{9} + \frac{1}{36}$

$\frac{1}{36} + \frac{1}{100} + \frac{1}{225} +$ &c. Et ſimiliter cum Serie in qua Denominatores ſunt quadrata numerorum naturalium comparari poſſunt eæ ubi Denominatores ſunt quadrata Pyramidalium, Triangulo-Triangularium, &c. Atque in genere poſſunt comparari omnes Series quæ definiuntur Æquationibus ſequentibus

$$T' = \frac{zz+a}{z+b \,.\, z+c} T, \quad T' = \frac{zz+a}{z+b+1 \,.\, z+c+1} T, \quad T' = \frac{zz+a}{z+b+2 \,.\, z+c+2} T, \text{ \&c.}$$

Ubi numerator manet idem, Denominatores autem formantur ſcribendo continue $z+1$ pro z.

Propositio X.

Invenire valorem Seriei quam proxime, quæ definitur Æquatione hujus formæ

$$T \times \overline{z^l + az^{l-1} + bz^{l-2} + \text{\&c.}} = rT' \times \overline{z^l + cz^{l-1} + dz^{l-2} + \text{\&c.}}$$

Casus I.

Primo ſit $r = 1$, & fingamus Æquationem ad Seriem eſſe $T' = \frac{z-m}{z} \times \frac{z-n+1}{z-n} T$. Ponamus eſſe $S = \frac{z+p}{q} T$ quam proxime, vel $S = \frac{z+p}{q} T + S2$ accurate: & progrediendo ad valores indeterminatarum ſuccedaneos, erit $S - T = \frac{z+p+1}{q} T' + S2 - T2$; harum Æquationum differentia dabit $T = \frac{z+p}{q} T - \frac{z+p+1}{q} T' + T2$; unde $T2 = \frac{z+p+1}{q} T' - \frac{z+p-q}{q} T$. Pro T' ſubſtitue ſuum valorem $\frac{z-m}{z} \times \frac{z-n}{z-n+1} T$, atque proveniet $T2 = \frac{z+p+1}{q} \times \frac{z-m}{z} \times \frac{z-n}{z-n+1} T - \frac{z+p-q}{q} T$: qui reductus ad communem Denominatorem dat

$T2 =$

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$$T2 = \frac{\overline{q-m} \times zz \left.\begin{matrix} + \overline{m-q} \times \overline{n-1} \\ - n - p \times \overline{m+1} \end{matrix}\right\} z + m.n.\overline{p+1}}{q.z.z-n+1} T,$$

Jam quo minor eſt Summa S2, eo proprius quantitas $\frac{z+p}{q}$T accedet ad valorem ipſius S; & eo minor erit S2, quo minor eſt Terminus ejus primus T2: is vero evadet minimus ubi variabilis z eſt minimarum dimenſionum in Numeratore ejus valoris; nam hic z magna ſupponitur: ponantur ergo Coefficientes dignitatum z^2 & z nihilo æquales, & habebuntur duæ Æquationes $q-m=0$, atque $\overline{m-q} \times \overline{n-1} - n - p \times \overline{m+1} = 0$, ad determinandum duas quantitates aſſumptas p, q. Prima dat $q=m$, ex quâ & ſecundâ eruitur $p = -\frac{n}{m+1}$. Unde erit $T2 = \frac{n.\overline{m-n+1}}{m+1.z.z-n+1}$ T, atque fere $S = \frac{\overline{m+1}.\overline{z-n}}{m.\overline{m+1}}$T. Q. E. I.

Et eadem prorſus ratione invenietur approximatio, ubi Æquatio ad Seriem eſt $T' = \frac{zz+az+b}{zz+cz+d}$ T, vel magis compoſita; ſed Coefficientes poſt b, & d in hoc computo non ingrediuntur.

CASUS II.

Sit jam Æquatio $T \times \overline{zz+az+b} + r T' \times \overline{zz+cz+d} = 0$, exiſtente r numero quolibet, exceptâ unitate. Finge $S = p \times \frac{z+m}{z+n} T + S2$; dein ſcribe valores variabilium poſteriores pro prioribus, & emerget $S - T = p \times \frac{z+m+1}{z+n+1} T' + S2 - T2$, quæ ablata de priore relinquit $T = p \times \frac{z+m}{z+n} T - p \times \frac{z+m+1}{z+n+1} T' + T2$: hæc Æquatio dabit $T2 = \frac{z - pz - mp + n}{z+n} T + p \times \frac{z+m+1}{z+n+1} T'$: ſed per Æquationem ad Seriem S, eſt $T' = -\frac{T}{r} \times \frac{zz+az+b}{zz+cz+d}$; quo ſcripto pro T', prodit

T2 =

$T2 = \frac{z - pz - mp + n}{z + n} T - \frac{pT}{r} \times \frac{z + m + 1}{z + n + 1} \times \frac{zz + az + b}{zz + cz + d}$. Si hujus valoris membra reducantur ad communem Denominatorem, & ponantur Coefficientes trium altiſſimarum dignitatum ipſius z æquales nihil ; Æquatio prima dabit $rp + p = r$, ſecunda $\overline{m - n} \times \overline{r + 1} = c - a$, tertia denique $\overline{c - a} \times \overline{2n + 1} + d - b = \overline{m - n} \times \overline{rc + rn + r + n} + ma$: & hæ tres Æquationes dant $p = \frac{r}{r + 1}$, $m = c - \frac{b - d}{a - c} - \frac{1}{r + 1}$, & $n = m + \frac{a - c}{r + 1}$. Dantur itaque Quantitates aſſumptæ p, m, & n; atque exinde etiam Quantitas $p \times \frac{z + m}{z + n} T$ quæ proxime æqualis eſt Seriei S. Q. E. I.

Propositio XI.

Si $T' = \frac{z - m}{z} \times \frac{z - n}{z - n + 1} T$ *ſit Æquatio exhibens relationem Terminorum Seriei* S, *pone*

$$T2 = \frac{1}{m} \times \frac{m}{m + 1} \times \frac{n}{z} \times \frac{m - n + 1}{z - n + 1} T,$$

$$T3 = \frac{2}{m + 2} \times \frac{m + 1}{m + 3} \times \frac{n + 1}{z + 1} \times \frac{m - n + 2}{z - n + 2} T2,$$

$$T4 = \frac{3}{m + 4} \times \frac{m + 2}{m + 5} \times \frac{n + 2}{z + 2} \times \frac{m - n + 3}{z - n + 3} T3,$$

$$T5 = \frac{4}{m + 6} \times \frac{m + 3}{m + 7} \times \frac{n + 3}{z + 3} \times \frac{m - n + 4}{z - n + 4} T4,$$

$$T6 = \frac{5}{m + 8} \times \frac{m + 4}{m + 9} \times \frac{n + 4}{z + 4} \times \frac{m - n + 5}{z - n + 5} T5,$$

&c.

Eritque $S = \frac{\overline{m + 1} . z - 1 . n}{m . \overline{m + 1}} T +$

$$\frac{\overline{m + 3} . \overline{z + 2} - 2 . \overline{n + 1}}{\overline{m + 2} . \overline{m + 3}} T2 +$$

$$\frac{\overline{m + 5} . \overline{z + 4} - 3 . \overline{n + 2}}{\overline{m + 4} . \overline{m + 5}} T3 +$$

$$\frac{\overline{m + 7} . \overline{z + 6} - 4 . \overline{n + 3}}{\overline{m + 6} . \overline{m + 7}} T4 +$$

$$\frac{\overline{m + 9} . \overline{z + 8} - 5 . \overline{n + 4}}{\overline{m + 8} . \overline{m + 9}} T5 +$$

&c.

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Per Propoſitionem præcedentem quantitas $\frac{\overline{m+1}\,\overline{z-n}}{m.\overline{m+1}}T$ proxime æqualis eſt Seriei S. Sit ergo $S = \frac{\overline{m+1}.\overline{z-n}}{m.\overline{m+1}}T + S_2$ accurate, & invenies $T_2 = \frac{1}{m} \times \frac{m}{m+1} \times \frac{n}{z} \times \frac{m-n+1}{z-n+1}T$, atque $T'_2 = \frac{z-m}{z+1} \times \frac{z-n}{z-n+2}T_2$ Æquationem ad Terminos Seriei S_2: ex qua data, per Propoſitionem priorem eruitur quantitas $\frac{\overline{m+3}.\overline{z+2}.-2\overline{n+1}}{m+2.m+3}T_2$ proxime æqualis Seriei S_2. Dein aſſumendo $S_2 = \frac{\overline{m+3}.\overline{z+2}-2.\overline{n+1}}{m+2.m+3}T_2 + S_3$, invenietur $T_3 = \frac{2}{m+2} \times \frac{m+1}{m+3} \times \frac{n+1}{z+1} \times \frac{m-n+1}{z-n+1}T_2$; & $T'_3 = \frac{z-m}{z+2} \times \frac{z-n}{z-n+3}T_3$ Æquatio ad Terminos Seriei S_3: unde comperies quantitatem $\frac{\overline{m+5}.\overline{z+4}-3.\overline{n+2}}{m+4.m+5}T_3$ proxime æqualem Seriei S_3. Et ſic porro proceditur. Igitur primus Terminus in valore ipſius S proxime æquatur ipſi S; atque Terminus ſecundus proxime æquatur ipſi S_2; tertius vero proxime æquatur ipſi S_3; & ita in reliquis: hoc eſt, Terminus primus proxime æqualis eſt Seriei cujus valor quæritur; ſecundus proxime æqualis eſt defectui primi Termini a vero; tertius proxime æqualis eſt defectui duorum primorum a vero; quartus proxime æqualis eſt defectui trium primorum a vero; & ſic deinceps. Igitur valor Summæ S eſt verus & celerrime convergens. Q. E. D.

Corollarium. Abrumpet valor Seriei S hic exhibitus, ubi n vel $m-n+1$ eſt nihil aut numerus integer & negativus: atque in aliis caſibus abibit in infinitum ad veritatem quam celerrime accelerans præterquam ubi, propter m nihil aut negativam, valor Seriei eſt infinite magnus.

Exemplum I.

Quæratur valor Seriei $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} +$ &c. Æquatio definiens relationem Terminorum eſt $T' = \frac{z-1}{z} \times \frac{z-1}{z}T$, exiſtentibus

tibus 2, 3, 4, 5, &c. valoribus indeterminatæ z. Hæc autem collata cum Æquatione in Theoremate dat $m=1$, $n=1$; quibus ſcriptis prodeunt

$$T2=\frac{T}{2z^{1}},\ T3=\frac{8T2}{6.\overline{z+1}^{1}},\ T4=\frac{27T3}{10.\overline{z+2}^{1}},\ T5=\frac{64T4}{14.\overline{z+3}^{1}},\ \&c.$$

$$\text{Et } S=\frac{2z-1}{2}T+\frac{2z+2}{6}T2+\frac{2z+5}{10}T3+\frac{2z+8}{14}T4+\frac{2z+11}{18}T5 + \&c.$$

Collige jam decem Terminos initiales, eorumque aggregatum reperies 1.5497.6773.1166.5406.9. Tum pro z ſcribe ejus valorem decimum primum, ſive 12; & Terminum decimum primum $\frac{1}{121}$ pro T; atque per computum invenies

T =	.0082.6446.2809.9173.55	.0950.4132.2314.0495.8
T2 =	- - - 2869.6051.4233.24	1.2434.9556.1677.4
T3 =	- - - - 22.6398.8277.97	65.6556.6006.1
T4 =	- - - - - - 3118.7593.62	7128.5928.3
T5 =	- - - - - - - 63.3652.70	123.2102.5
T6 =	- - - - - - - - 1.7188.93	2.9690.0
T7 =	- - - - - - - - - 583.96	920.9
T8 =	- - - - - - - - - - 23.78	34.9
T9 =	- - - - - - - - - - - 1.12	1.5
	S =	.0951.6633.5681.6857.4

Ex hoc computo habemus S = .0951.6633.5681.6857.4, quæ adjecta aggregato initialium, exhibet 1.6449.3406.6848.2264.3 pro valore Seriei propoſitæ.

Exemplum II.

Summanda ſit Series $1+\frac{1.1}{2.3}A+\frac{3.3}{4.5}B+\frac{5.5}{6.7}C+\frac{7.7}{8.9}D+\frac{9.9}{10.11}E+\&c.$

Æquatio eandem denotans eſt $T'=\frac{z-\frac{1}{2}}{z}\times\frac{z-\frac{1}{2}}{z+\frac{1}{2}}T$, uti patebit ſcribendo valores 1, 2, 3, 4, &c. ſucceſſive pro z. Æquatio autem in Theoremate collata cum hac, dat $m=\frac{1}{2}$, $n=\frac{1}{2}$, adeoque

T2 =

$$T2 = \frac{2.1.1}{1.3.z.\overline{2z+1}} T, \quad T3 = \frac{2.4.9}{5.7.\overline{z+1}.\overline{2z+3}} T2, \quad T4 = \frac{2.9.25}{9.11.\overline{z+2}.\overline{2z+5}} T3,$$
&c.

$$\text{Et } S = \frac{6.z - 1.2}{1.3} T + \frac{14.\overline{z+1} - 3.4}{5.7} T2 + \frac{22.\overline{z+2} - 5.6}{9.11} T3 + \&c.$$

Per additionem reperies Summam decem Terminorum initialium eſſe 1.3916.94645943.2880.5; dein ut eruatur Summa reliquorum ſcribe 11 pro z, & Terminum decimum primum pro T, & habebis

T =	.0083.9003.5809.6168.15	.1789.9383.0605.1587.3
T2 =	2210.8921.7644.71	1.0738.6191.4274.3
T3 =	15.1604.0349.56	45.9406.1665.3
T4 =	1963.2742.16	4570.9051.0
T5 =	38.8835.92	76.0818.3
T6 =	1.0484.94	1.8104.4
T7 =	358.18	562.5
T8 =	14.77	21.5
T9 =	71	1.0
		S = .1791.0168.0851.6085.6

Addatur nunc S aggregato initialium, & prodibit 1.5707.9632.6794.8966.1 pro valore Seriei, id eſt, pro ſemiperiferia Circuli, cujus Diameter eſt unitas.

Exemplum III.

Proponatur jam Series $\frac{1}{3} + \frac{1.3}{2.7} A + \frac{3.7}{4.11} B + \frac{5.11}{6.15} C + \frac{7.15}{8.19} D$ &c. quæ definitur Æquatione $T' = \frac{z - \frac{1}{2}}{z} \times \frac{z - \frac{1}{4}}{z + \frac{1}{4}} T$, in qua 1, 2, 3, 4, &c. ſunt valores indeterminatæ z. Et erit $m = \frac{1}{2}$, $n = \frac{1}{4}$; & propterea

$$T2 = \frac{5.1.1}{6.z.\overline{4z+3}} T, \quad T3 = \frac{9.2.3}{14.\overline{z+1}.\overline{4z+7}} T2, \quad T4 = \frac{13.3.5}{22.\overline{z+2}.\overline{4z+11}} T3,$$
&c.

$$\text{Et } S = \frac{6.z - 1.1}{1.3} T + \frac{14.\overline{z+2} - 2.5}{5.7} T2 + \frac{22.\overline{z+4} - 3.9}{9.11} T3 + \&c.$$

Colligo

Colligo jam novem Terminos initiales, quorum Summa prodit .5055.0041.4718.3195.8, & ſcribens 10 pro z, decimum vero Terminum pro T, habeo

T =	.0047.5565.5924.4791.67	.0935.2789.9848.0902.9
T2 =	921.6387.4505.41	4160.5406.2053.0
T3 =	6.8760.0058.48	19.5167.2893.3
T4 =	995.8557.00	2185.7755.9
T5 =	22.0991.75	40.9826.8
T6 =	6653.41	1.0937.5
T7 =	252.54	379.0
T8 =	11.51	16.1
T9 =	61	8
		S = .0935.6970.2649.4765.3

Summa denique initialium addita ipſi S, conficit .5990.7011.7367.7961.1 pro valore Seriei, id eſt, pro Ordinata Curvæ Elaſticæ. Et hunc numerum *Jacobus Bernoullius* recte invenit contineri intra limites .5983 & .6004.

Exemplum IV.

Proponatur Series $1 + \frac{1.1}{2.5}A + \frac{3.5}{4.9}B + \frac{5.9}{6.13}C + \frac{7.13}{8.17}D + \frac{9.17}{10.21}E +$ &c. quæ definitur Æquatione $T' = \frac{z - \frac{1}{2}}{z} \times \frac{z - \frac{3}{4}}{z + \frac{1}{4}} T$, exiſtentibus 1, 2, 3 4, &c. valoribus indeterminatæ ſuccedentibus. Eſt vero $m = \frac{1}{2}$, $n = \frac{3}{4}$; & proinde

$$T2 = \frac{1.1.3}{2.z.\overline{4z+1}} T,\quad T3 = \frac{3.2.7}{10.\overline{z+1}.\overline{4z+5}} T2,\quad T4 = \frac{5.3.11}{18.\overline{z+2}.\overline{4z+9}} T3,$$
&c.

$$\text{Et } S = \frac{2z-1}{1} T + \frac{2z+2}{5} T2 + \frac{2z+5}{9} T3 + \frac{2z+8}{13} T4 + \frac{2z+11}{17} T5 + \text{\&c.}$$

Summa novem Terminorum initialium eſt 1.2157.0599.7306.1360.6. Et ut habeatur Summa reliquorum, pone 10 pro z, & decimum Terminum pro T, ac per computum obtinebis

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T = .0050.1271.8406.8834.46	.0952.4164.9730.7854.7
T2 = - - - 1833.9213.6837.20	8069.2540.2083.7
T3 = - - - - 15.5605.4494.38	43.2237.3595.5
T4 = - - - - - 2425.8219.16	5224.8472.0
T5 = - - - - - - 56.8742.44	103.7118.6
T6 = - - - - - - - 1.7922.51	2.9017.4
T7 = - - - - - - - - 707.95	1047.8
T8 = - - - - - - - - - 33.45	46.1
T9 = - - - - - - - - - - 1.83	2.3
	S = .0953.2277.9839.9238.1

Adjiciatur jam S aggregato initialium, & obtinebis 1.3110.2877.7146.0598.7 pro valore Seriei, id eſt, pro longitudine Curvæ Elaſticæ, modo in lineam rectam extenſa foret. Hunc autem numerum determinavit *Bernoullius* conſiſtere inter limites 1.308 & 1.315. Quod ſi longitudini Elaſticæ adjiciatur ſua Ordinata, habebitur numerus 1.9100.9889.4513.8559.8 qui eſt ſemiperiferia Ellipſeos habentis 1 & $\sqrt{2}$ pro Axibus. Et hæc Exempla ſufficiant; haud enim immoror Seriebus quæ per hanc Propoſitionem ſummari poſſint accurate.

Scholion.

Hoc Theorema nullo fere negotio exhibet Areas Curvarum binomialium quarum Ordinatæ continentur ſub forma $x^{\theta} \times \overline{e+fx^{\eta}}^{\lambda}$, in unico tamen caſu, ſcilicet ubi eſt $e+fx^{\eta}=0$, ſive $x^{\eta}=-\frac{e}{f}$; hoc eſt in eo caſu in quo Series pro Area convergit lentiſſime. Sed ubi Areæ non producendæ ſunt ultra octo aut novem figuras, ſufficit quærere Summam quatuor Terminorum initialium, nam S dabit eam reliquorum exiguo labore; imo ſi nulli Termini initiales colligantur, ſed inchoetur tranſmutatio ad primum Terminum, valor ipſius S approximabit ad valorem totius Seriei ſat celeriter. Series autem in Theoremate, redditur generalior, atque extenditur ad caſus qui ad Quadraturas non attinent, ut ſequitur. Sit Æquatio ad Seriem $T' = \frac{zz+m}{zz+nz+r}T$, & ponatur

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$a = n - \frac{r - m}{n}$

$b = n + 2 - \frac{r - m + n + 1}{n + 2}$,

$c = n + 4 - \frac{r - m + 2n + 4}{n + 4}$,

$d = n + 6 - \frac{r - m + 3n + 9}{n + 6}$,

$e = n + 8 - \frac{r - m + 4n + 16}{n + 8}$,

&c.

$T2 = \frac{T}{n - 1} \times \frac{ma + \overline{n - a} \times r}{zz + nz + r}$,

$T3 = \frac{T2}{n + 1} \times \frac{mb + \overline{n - b + 2} \times \overline{r + n + 1}}{zz + \overline{n + 2}.z + r + n + 1}$,

$T4 = \frac{T3}{n + 3} \times \frac{mc + \overline{n - c + 4} \times \overline{r + 2n + 4}}{zz + \overline{n + 4}.z + r + 2n + 4}$,

$T5 = \frac{T4}{n + 5} \times \frac{md + \overline{n - d + 6} \times \overline{r + 3n + 9}}{zz + \overline{n + 6}.z + r + 3n + 9}$,

$T6 = \frac{T5}{n + 7} \times \frac{me + \overline{n - e + 8} \times \overline{r + 4n + 16}}{zz + \overline{n + 8}.z + r + 4n + 16}$,

&c.

Eritque S =

$$\frac{z + a - 1}{n - 1} T + \frac{z + b - 1}{n + 1} T2 + \frac{z + c - 1}{n + 3} T3 + \frac{z + d - 1}{n + 5} T4 + \frac{z + e - 1}{n + 7} T5 + \&c.$$

EXEMPLUM.

Proponatur Series $\frac{1}{2} + \frac{1.1}{2.4} A + \frac{3.3}{4.6} B + \frac{5.5}{6.8} C + \frac{7.7}{8.10} D + \frac{9.9}{10.12} E$ &c. quæ definitur Æquatione $T' = \frac{z}{z + \frac{1}{2}} \times \frac{z}{z + \frac{3}{2}} T$, exiſtentibus $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, &c. valoribus Abſciſſæ z ſucceſſivis. Hæc Æquatio comparari nequit cum $T' = \frac{z - m}{z} \times \frac{z - n}{z - n + 1} T$, eâ ſcilicet in Propoſitione; in qua utique ſunt duo factores $z - n$, & $z - n + 1$ unitate differentes, quorum alter eſt in Numeratore, alter in Denominatore: quum tamen in Æquatione propoſitam Seriem definiente differentiæ factorum in Numeratore & Denominatore ſint $\frac{1}{2}$ & $\frac{3}{2}$. Igitur multiplico factores in ſe, & prodit $T' = \frac{zz}{zz + 2z + \frac{3}{4}} T$, & pergo ad Æquationem in Scholio; quam cum altera comparans, habeo $m = 0$, $n = 2$, & $r = \frac{3}{4}$; & hiſce ſcriptis oriuntur

$a =$

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$a = \frac{13}{8}$,	$T2 = \frac{1}{8} \times \frac{9}{4zz+8z+3}\, T$,
$b = \frac{49}{16}$,	$T3 = \frac{3}{16} \times \frac{25}{4zz+16z+15}\, T2$,
$c = \frac{109}{24}$,	$T4 = \frac{5}{24} \times \frac{49}{4zz+24z+35}\, T3$,
$d = \frac{193}{32}$,	$T5 = \frac{7}{32} \times \frac{81}{4zz+32z+63}\, T4$,
$e = \frac{301}{40}$,	$T6 = \frac{9}{40} \times \frac{121}{4zz+40z+99}\, T5$,
&c.	&c.

$$\text{Et } S = \frac{8z+5}{1.8}\, T + \frac{16z+33}{3.16}\, T2 + \frac{24z+85}{5.24}\, T3 + \frac{32z+161}{7.32}\, T4 + \&c.$$

Quære jam aggregatum ſex Terminorum initialium, idemque prodibit .6106.6818.2373.0. Tum ſubſtitue ſeptimum Terminum pro T, & valorem ſeptimum $\frac{13}{2}$ pro z; & per calculum invenies

T =	.0036.3492.9656.98	.0258.9887.3806.0
T2 =	1825.5785.11	5210.5053.3
T3 =	29.7131.92	59.6739.9
T4 =	8425.62	1.3879.7
T5 =	339.30	491.0
T6 =	17.50	23.1
T7 =	1.09	1.3
		S = .0259.5158.9994.3

Inventa hoc modo S, eadem addatur Summæ initialium & prodibit .6366.1977.2367.3 pro valore Seriei propoſitæ. Si vero unitas dividatur per hunc numerum, quotus dabit Aream Circuli. Et eadem ratione tranſmutantur Series quæ definiuntur Æquatione hujuſmodi $T' = \frac{z^3+az^2+bz+c}{z^3+dz^2+ez+f}\, T$, vel etiam generaliori. Et notandum eſt Æquationem in Scholio $T' = \frac{zz+m}{zz+nz+r}\, T$, non ad pauciores caſus ideo extendi quod in Numeratore deſit membrum in quo z eſt unius dimenſionis: illud enim ſemper tolli poteſt mutando principium Abſciſſæ, eoque pacto Theorema redditur ſimplicius.

PROPO-

Propositio XII.

Oporteat tranſmutare Seriem definitam Æquatione hujus formæ

$$\mathrm{T}\times\overline{z^{\theta}+az^{\theta-1}+bz^{\theta-2}+\&c.}+r\mathrm{T}'\times\overline{z^{\theta}+cz^{\theta-1}+dz^{\theta-2}+\&c.}=0.$$

in aliam celerrime convergentem.

Aſſume $m=c-\frac{b-d}{a-c}-\frac{1}{r+1}$, $n=m+\frac{a-c}{r+1}$; atque per Propoſitionem decimam erit $\frac{r}{r+1}\times\frac{z+m}{z+n}\mathrm{T}$ proxime æqualis Seriei. Sit itaque quantitas illa primus Terminus Seriei tranſmutatæ: & ad inveniendum ſecundum, ponatur $\mathrm{S}=\frac{r}{r+1}\times\frac{z+m}{z+n}\mathrm{T}+\mathrm{S}2$; deinde per Propoſitionem nonam quæratur Æquatio ad Terminos Seriei S2; & ex illa eliciatur quantitas proxime æqualis Seriei S2, eodem prorſus modo ac approximatio ad Seriem S prius inventa fuit; & quantitas illa erit ſecundus Terminus Seriei tranſmutatæ: ac progrediendo ſuper hiſce veſtigiis, invenire licet Terminos ſequentes tot quot volueris. Q. E. I.

Exemplum I.

Tranſmutanda ſit Series $\sqrt{12}$ in $1-\frac{1}{3.3}+\frac{1}{5.9}-\frac{1}{7.27}+\frac{1}{9.81}-\frac{1}{11.243}+$ &c. quam adhibuit *Halleius* in Quadratuta Circuli. Relatio Terminorum definitur Æquatione $z\mathrm{T}+3\mathrm{T}'\overline{z+1}=0$, in qua valores ipſius z ſunt $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, &c. Comparando autem hanc cum Æquatione in Problemate, erit $r=3$, $a=0$, $b=0$, $c=1$, $d=0$; & inde $m=\frac{3}{4}$, $n=\frac{1}{2}$; quibus ſcriptis habebitur $\frac{3}{4}\times\frac{4z+3}{4z+2}\mathrm{T}$ pro primo Termino Seriei tranſmutatæ.

Finge $\mathrm{S}=\frac{3}{4}\times\frac{4z+3}{4z+2}\mathrm{T}+\mathrm{S}2$, tum ſcribendo valores variabilium ſuc-

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cedentes pro antecedentibus, proveniet $S - T = \frac{3}{4} \times \frac{4z+7}{4z+6} T' + S2 -$ $T2$, quarum Æquationum differentia dabit $T = \frac{3}{4} \times \frac{4z+3}{4z+2} T - \frac{3}{4} \times$ $\frac{4z+7}{4z+6} T' + T2$, & exinde $T2 = \frac{1}{4} \times \frac{4z-1}{4z+2} T + \frac{3}{4} \times \frac{4z+7}{4z+6} T'$. Sed per Æquationem ad Seriem eſt $T' = -\frac{1}{3} T \times \frac{z}{z+1}$; quo ſubſtituto, obtinebis $T2 = -\frac{1}{8} \times \frac{1}{2z+1} \times \frac{2}{2z+2} \times \frac{3}{2z+3} T$. In quo valore ſcribendo quantitates variabiles poſteriores pro prioribus, emerget $T'2 =$ $-\frac{1}{8} \times \frac{1}{2z+3} \times \frac{2}{2z+4} \times \frac{3}{2z+5} T'$, vel ſubſtituendo $-\frac{1}{3} T \times \frac{z}{z+1}$ pro T', eveniet $T'2 = +\frac{1}{8} \times \frac{1}{2z+3} \times \frac{2}{2z+4} \times \frac{1}{2z+5} \times \frac{z}{z+1} T$; per hunc valorem ipſius $T'2$, eſt $T = \frac{2}{z} \times \overline{2z+2} \times \overline{2z+3} \times \overline{2z+4} \times \overline{2z+5}\, T'2$: Et per valorem ipſius $T2$ antea repertum eſt $T = -\frac{4}{3} \times \overline{2z+1} \times \overline{2z+2} \times$ $\overline{2z+3}\, T2$. Denique æquando hoſce duos valores, Termini T, prodibit $\overline{zz+\frac{1}{2}z} \times T2 + 3T'2 \times \overline{zz+\frac{9}{2}z+5} = 0$, Æquatio utique ad Terminos Seriei $S2$; quæ collata cum illa in Propoſitione dat $r = 3$, $a = \frac{1}{2}$, $b = 0$, $c = \frac{9}{2}$, $d = 5$; hinc fit $m = 3$, $n = 2$; & proinde erit $\frac{3}{4} \times \frac{z+3}{z+2} T2$ proxime æqualis Seriei $S2$; adeoque Terminus ſecundus Seriei tranſmutatæ.

Ad inveniendum tertium aſſumatur $S2 = \frac{3}{4} \times \frac{z+3}{z+2} T2 + S3$, & inde per methodum Differentialem erit $S2 - T2 = \frac{3}{4} \times \frac{z+4}{z+3} T'2 + S3$ $- T3$, qua de priore ſubducta, manet $T2 = \frac{3}{4} \times \frac{z+3}{z+4} T2 - \frac{3}{4} \times \frac{z+4}{z+3} T'2$ $+ T3$, ex qua oritur $T3 = \frac{1}{4} \times \frac{z-1}{z+2} T2 + \frac{3}{4} \times \frac{z+4}{z+3} T'2$. Sed per

Æqua-

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Æquationem ad Seriem S2, eſt $T'2 = -\frac{1}{3} \times \frac{z}{z+2} \times \frac{2z+1}{2z+5} T2$; quo ſubſtituto in valore ipſius T3, provenit $T3 = -\frac{1}{8} \times \frac{4}{2z+4} \times \frac{5}{2z+5} \times \frac{6}{2z+6} T2$: ubi ſcribendo valores variabilium conſequentes pro antecedentibus, orietur $T'3 = -\frac{1}{8} \times \frac{4}{2z+6} \times \frac{5}{2z+7} \times \frac{6}{2z+8} T'2$; vel ſubſtituendo pro T′2 ſuum valorem, erit $T'3 = \frac{1}{8} \times \frac{4}{2z+6} \times \frac{5}{2z+7} \times \frac{6}{2z+8} \times \frac{1}{3} \times \frac{z}{z+2} \times \frac{2z+1}{2z+5} T2$. Per valores Terminorum T3, & T′3, ſi eliminetur T2, reſultabit $\overline{zz+\frac{1}{2}z}\, T3 + 3T'3 \times \overline{zz+\frac{15}{2}z+14} = 0$, quæ eſt Æquatio ad Terminos Seriei S3; eaque comparata cum illa in Theoremate exhibet $a = \frac{1}{2}$, $b = 0$, $c = \frac{15}{2}$, $d = 14$; adeoque $m = \frac{21}{4}$, $n = \frac{14}{4}$; & propterea $\frac{3}{4} \times \frac{4z+21}{4z+14} T3$ erit approximatio ad Seriem S3, ſive Terminus tertius Seriei tranſmutatæ.

Et ſimili proceſſu, ponendo $S3 = \frac{3}{4} \times \frac{4z+21}{4z+14} T3 + S4$, invenies Terminum quartum eſſe $\frac{3}{4} \times \frac{4z+30}{4z+20} T4$; exiſtente $T4 = -\frac{1}{8} \times \frac{7}{2z+7} \times \frac{8}{2z+8} \times \frac{9}{2z+9} T3$. Atque eadem omnino ratione Series tranſmutata ad libitum producetur. Sed jam Terminorum progreſſio ap aret, exiſtentibus

$$T2 = -\frac{1}{8} \times \frac{1}{2z+1} \times \frac{2}{2z+2} \times \frac{3}{2z+3} T,$$

$$T3 = -\frac{1}{8} \times \frac{4}{2z+4} \times \frac{5}{2z+5} \times \frac{6}{2z+6} T2,$$

$$T4 = -\frac{1}{8} \times \frac{7}{2z+7} \times \frac{8}{2z+8} \times \frac{9}{2z+9} T3,$$

$$T5 = -\frac{1}{8} \times \frac{10}{2z+10} \times \frac{11}{2z+11} \times \frac{12}{2z+12} T4,$$

&c.

Ac

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$$\text{Ac } S = \frac{3}{4} \text{ in } \frac{4z+3}{4z+2}T + \frac{4z+12}{4z+8}T2 + \frac{4z+21}{4z+14}T3 + \frac{4z+30}{4z+20}T4 + \&c.$$

Summa decem Terminorum initialium eſt 3.1415.9051.0938.0800.9964.2 ; & decimus primus ſubſtitutus pro T, ac $\frac{21}{2}$ pro z, dant

$+T =$	.0000.0279.3565.0014.1347.8
$-T2 =$	172.5274.8297.5
$+T3 =$	1474.5938.7
$-T4 =$	3.8136.0
$+T5 =$	192.2
$-T6 =$	1.5

.0000.0214.2791.3363.1147.5	.0000.0000.0139.7472.6106.4
1244.1885.8	3.3215.2
171.7	1.4
+ .0000.0214.2791.4607.3205.0	− .0000.0000.0139.7475.9323.0

Subduc jam Summam negativorum ab ea affirmativorum, & proveniet .0000.0214.2651.7131.3882.0, quæ addita aggregato initialium exhibet 3.1415.9265.3589.7932.3846.2 pro valore Seriei propoſitæ. Et hi ſex Termini Seriei tranſmutatæ tantundem efficiunt ac trigintaduo Seriei ſimplicis. Sed horum Theorematum uſus latius patet in Seriebus lentiſſime convergentibus, ad quarum utique valores pervenire non licet per meram additionem Terminorum.

Exemplum II.

Tranſmutanda ſit Series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} +$ &c. cujus Æquatio eſt $zzT + \overline{zz+2z+1}\,T' = 0$, exiſtentibus 1, 2, 3, 4, 5, &c. valoribus ipſius z ordine ſuccedentibus. Hic erit $a = 0$, $b = 0$, $c = 2$, $d = 1$, $r = 1$; unde $m = 1$, $n = 0$, & propterea $\frac{1}{2} \times \frac{z+1}{z} T$ eſt primus Terminus Seriei tranſmutatæ.

Ut eruatur ſecundus, finge $S = \frac{1}{2} \times \frac{z+1}{z} T + S2$; & per methodum Differentialem prodibit $S - T = \frac{1}{2} \times \frac{z+2}{z+1} T' + S2 - T2$; quæ deducta

deducta de priore relinquit $T = \frac{1}{2} \times \frac{z+1}{z} T - \frac{1}{2} \times \frac{z+2}{z+1} T' + T2$; ex quâ invenietur $T2 = \frac{1}{2} \times \frac{z-1}{z} T + \frac{1}{2} \times \frac{z+2}{z+1} T'$: ſed Æquatio ad Seriem S dat $T' = \frac{-zzT}{zz+2z+1}$, quo ſubſtituto, proveniet $T2 = -\frac{2z+1}{2z} \times \frac{T}{\overline{z+1}|^3}$. Scribe jam valores variabilium conſequentes pro antecedentibus, & orietur $T'2 = -\frac{2z+3}{2z+2} \times \frac{T}{\overline{z+2}|^3}$, vel rurſus ponendo pro T' ſuum valorem, $T'2 = \frac{1}{2} \times \frac{2z+3}{\overline{z+1}|^3} \times \frac{zz}{\overline{z+2}|^3} T$. Per valores Terminorum $T2$ & $T'2$, extermina T, & habebis $\overline{z^4 + \frac{3}{2} z^3} \times T2 + \overline{z^4 + \frac{13}{2} z^3 + 15z^2 + 14z + 4} \times T'2$, quæ eſt Æquatio ad Terminos Seriei S2: eaque collata cum illa in Propoſitione, dat $a = \frac{3}{2}$, $b = 0$, $c = \frac{13}{2}$, $d = 15$, $r = 1$; & proinde $m = 3$, $n = \frac{1}{2}$; quapropter eſt $\frac{z+3}{2z+1} T2$ valor Seriei S2 quamproxime, ſive Terminus ſecundus Seriei tranſmutatæ. Atque ſuper hiſce veſtigiis progrediendo ut in Exemplo ſuperiore, invenies

$$T2 = -\frac{2z+1}{2z} \times \frac{T}{\overline{z+1}|^3},\ T3 = -\frac{2z+2}{2z+1} \times \frac{8\,T2}{\overline{z+2}|^3},\ T4 = -\frac{2z+3}{2z+2} \times \frac{27\,T3}{\overline{z+3}|^3}, \&c.$$

$$\text{Ac } S = \frac{z+1}{2z} T + \frac{z+3}{2z+1} T2 + \frac{z+5}{2z+2} T3 + \frac{z+7}{2z+3} T4 + \&c.$$

Vel ob faciliorem computum, pone

$$A = T,\ B = \frac{A}{\overline{z+1}|^3},\ C = \frac{8B}{\overline{z+2}|^3},\ D = \frac{27C}{\overline{z+3}|^3},\ E = \frac{64D}{\overline{z+4}|^3}, \&c.$$

$$\text{Eritque } S = \frac{1}{2z} \text{ in } \overline{z+1}\,A - \overline{z+3}\,B + \overline{z+5}\,C - \overline{z+7}\,D + \overline{z+9}\,E - \&c.$$

Summa decem Terminorum Seriei ſummandæ ſub propriis ſignis eſt .8179.6217.5610.9851.3. Tum ut habeatur Summa reliquorum ſubſtitue Terminum decimum primum, hoc eſt, $\frac{1}{121}$ pro T, & pro z valorem ſuum reſpectivum, ſcilicet 11; ac habebis

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$$A=\frac{1}{121},\ B=\frac{1.1.1}{12.12.12}A,\ C=\frac{2.2.2}{13.13.13}B,\ D=\frac{3.3.3}{14.14.14}C, \&c.$$

$$\text{Et } S=\frac{1}{11} \text{ in } 6A-7B+8C-9D+10E-\&c.$$

Computus autem ſic ſe habet

A =	.0082.6446.2809.9173.5
B =	478.2675.2372.2
C =	1.7415.2944.5
D =	171.3604.0
E =	3.2495.0
F =	991.7
G =	43.6
H =	2.6
I =	.2

Deinde

.0495.8677.6859.5041.0	.0000.3347.8726.6605.4
13.9322.3556.0	1542.2436.0
32.4950.0	1.0908.7
523.2	33.8
2.8	— .0000.3348.0296.9983.9
+ .0495.8691.6214.4073.0	

Differentia harum Summarum diviſa per 11, dat S = .0045.0485.7813.1280.8, quæ adjecta aggregato decem initialium, conficit .8224.6703.3424.1132.1 pro valore Seriei. Quumque hic numerus dimidium ſit illius in Exemplo primo Propoſitionis decimæ reperti; concludendum eſt utrumque computum recte inſtitutum fuiſſe. Etenim Series $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+$ &c. ubi Termini ſunt alternatim negativi & affirmativi, dimidium eſt hujus $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+$ &c. in qua omnes ſunt ejuſdem ſigni.

Exemplum III.

Proponatur Series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+$ &c. quæ definitur Æquatione $zT+\overline{z+1}\,T'=0$, in qua valores Abſciſſæ ſunt $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$,

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$\frac{5}{2}$, $\frac{7}{2}$, &c. Et ineundo computum juxta hanc Propoſitionem, devenietur ad regulam ſequentem. Pone $A = T$, deinde

$$B = \frac{1}{2} A \times \frac{1}{2z+1} \times \frac{1}{\overline{z+1}|^2},$$

$$C = \frac{1}{2} B \times \frac{2}{2z+3} \times \frac{9}{\overline{z+2}|^2},$$

$$D = \frac{1}{2} C \times \frac{3}{2z+5} \times \frac{25}{\overline{z+3}|^2},$$

$$E = \frac{1}{2} D \times \frac{4}{2z+7} \times \frac{49}{\overline{z+4}|^2}.$$

&c.

Eritque $S = \frac{1}{4z}$ in $\overline{2z+1}A - \overline{2z+5}B + \overline{2z+9}C - \overline{2z+13}D + \overline{2z+17}E -$ &c.

Summa duodecem Terminorum in Serie propoſita eſt .7646.0069.1481.8329.5, atque Terminus decimus tertius $\frac{1}{25} = T$, & $z = \frac{25}{2}$, quibus ſcriptis, habemus $A = \frac{1}{25}$,

$B = \frac{1.1.1}{13.27.27} A$, $B = \frac{2.3.3}{14.29.29} B$, $D = \frac{3.5.5}{15.31.31} C$, $E = \frac{4.7.7}{16.33.33} D$, &c.

ac $S = \frac{1}{25}$ in $13\,A - 15\,B + 17\,C - 19\,D + 21\,E -$ &c.

A =	.0400.0000.0000.0000.0
B =	422.0744.9614.9
C =	6452.6422.0
D =	33.5725.4
E =	3776.5
F =	73.4
G =	2.2
H =	1

.5200.0000.0000.0000.0	.0000.6331.1174.4223.5
10.9694.9174.0	637.8782.6
7.9306.5	1688.2
55.0	2.7
+ .5200.0010.9702.5835.5	— .0000.6331.1812.4697.0

Dein

Dein dividendo differentiam Summarum per 25, prodit S = .0207.9747.1915.6153.5, quæ una cum aggregato initialium conficit .7853.9816.3397.4483.0 pro valore Seriei ſummandæ.

SCHOLION.

Sicut una Æquatio definit Series numero infinitas, ita una Tranſmutatio inſervit Seriebus numero infinitis: Et unumquodque Exemplum pro Theoremate eſt habendum; ſic Tranſmutatio in ultimo Exemplo inſervit Seriei generali

$$\frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \frac{1}{m+4n} - \&c.$$

Obſervandum eſt legem continuandi Series per hanc Propoſitionem tranſmutatas, non ſemper ſe prodere ut in Exemplis quæ hic elegi: hoc vero neutiquam incommodabit opus. Nam poſtquam colliguntur ſex circiter Termini Seriei ſummandæ, tres vel quatuor Seriei tranſmutatæ dabunt quæſitum ſatis accurate pro uſibus quibuſcunque; in praxi enim raro opus eſt continuare computum ultra novem aut decem figuras. Atque eodem res redit, ſive Termini Seriei tranſmutandæ ſint iiſdem vel contrariis ſignis affecti, ſive ſint vel non ſint aſſignabiles. Labor enim ſemper erit levis, præterquam ubi in Æquatione ad Seriem, quantitas r eſt negativa & ſimul proxime æqualis unitati; ut ſi ſit $r = -\frac{9}{10}$, vel $= -\frac{99}{100}$, computus erit laborioſus. Sed hoſce caſus facile evitabit Analyſta peritus, quibus igitur remedium proferre non eſt operæ pretium.

Adjungerem quoque nonnulla de Seriebus hujuſmodi

$$x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \&c.$$

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \&c.$$

Ubi Termini in infinitum producti non habent datam rationem ad ſe invicem, ut in Seriebus de quibus hactenus egimus, ſed antecedentes ſunt infinite majores conſequentibus. Hæ exhibent Numerum ex dato Logarithmo, vel Sinum ex dato Arcu, & ſunt ſui generis ſimpliciſſimæ. Tranſmutari poſſunt per principia in ſuperioribus poſita; cæterum res commodius abſolvitur abſque Tranſmutationibus. Ut in Serie $x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \&c.$ quæ exhibet Numerum ex dato Logarithmo

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rithmo x, ſi eſſet $x = 12.3785$, rejicerem Characteriſticam 12, & quærerem Numerum Logarithmi .3785 qui prodiret in Serie celeriter convergente ob Logarithmum jam unitate minorem; quo dato non lateret Numerus Logarithmi 12.3785. Et eodem res redit ſive Logarithmus ſit Tabularis ſive Hyperbolicus.

Et ſimiliter in quærendo Sinum ex dato Arcu, ſi idem major ſit Quadrante, ſubducatur ex Semicirculo, & manebit Arcus Quadrante minor eundem habens Sinum ac prior, utique ejus ſupplementum ad Semicirculum. Arcus autem minor Quadrante dabit ſuum Sinum in Serie celeriter convergente.

Series quæ definiuntur Æquationibus quæ involvunt tres vel plures Terminos Seriei poſſunt ſummari accurate vel quamproxime ex Analyſi ſupra tradita: ſufficiat autem hujuſmodi computationum jeciſſe fundamentum & aperuiſſe viam aliis quibus eſt otium & animus hâc de materiâ plura ſcutari. Sed ne omnino neglecta videatur, dabimus Theorema generale ex principiis *Moivræi*, quod tam ad Summationem quam ad Tranſmutationem hujus generis Serierum extenditur.

Propositio XIII.

In Seriebus ex Diviſione ortis, eadem eſt relatio inter Terminos ac Summas ſucceſſivas.

Sit Fractio $\frac{1}{1-3x+xx}$, quæ in Seriem reſoluta eſt $1+3x+8x^2+21x^3+55x^4+$ &c. Tum Summæ ſucceſſivæ erunt

$$\frac{1}{1-3x+xx} = 1+3x+8x^2+21x^3+55x^4+144x^5+\&c.$$

$$\frac{3x-xx}{1-3x+xx} = 3x+8x^2+21x^3+55x^4+144x^5+\&c.$$

$$\frac{8x^2-3x^3}{1-3x+xx} = 8x^2+21x^3+55x^4+144x^5+\&c.$$

$$\frac{21x^3-8x^4}{1-3x+xx} = 21x^3+55x^4+144x^5+\&c.$$

$$\frac{55x^4-21x^5}{1-3x+xx} = 55x^4+144x^5+\&c.$$

&c.

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Et ſenſus Propoſitionis eſt quod hæ Summæ eodem numero ſumptæ eandem ubique habent relationem ac Termini Seriei totidem numero ſumpti. Exempli gratiâ, in caſu præſente relatio inter tres quoſvis Terminos ordine ſuccedentes erit $xxT - 3xT' + T'' = 0$: atque ea de cauſa relatio inter tres Summas ſucceſſivas erit etiam $xxS - 3xS' + S'' = 0$, ut experienti patebit. Propoſitio vero ſic demonſtratur.

Sint r, s, t datæ quantitates, & aſſumatur Æquatio ad Summas $rS + sS' + tS'' = 0$, dein ſubſtituendo valores variabilium ſuccedentes pro præſentibus, habebitur $rS' + sS'' + tS''' = 0$, quæ ſubducta de priori, relinquit $rS - rS' + sS' - sS'' + tS'' - tS''' = 0$; in hâc ſubſtitue T pro $S - S'$, T′ pro $S' - S''$, ac T″ pro $S'' - S'''$, & proveniet $rT + sT' + tT'' = 0$. Et hæc eſt eadem relatio ac ea Summarum primo aſſumpta. Et ſi plures vel pauciores ſint Summæ, eodem prorſus modo Propoſitio demonſtrabitur.

Corollarium. Hinc habemus methodum ſummandi haſce Series ex data relatione Terminorum; ut ſequentibus Exemplis fiet manifeſtum.

Exemplum I.

Detur Æquatio ad Terminos $rT + sT' = 0$, & per hanc Propoſitionem erit etiam eadem relatio Summarum, utique $rS + sS' = 0$; pro S′ ſubſtitue ſuum valorem $S - T$, & orietur $rS + sS - sT = 0$, unde fit $S = \frac{s}{r+s}T$. Quare datur Summa S ex dato ejus primo Termino T. Ut ſi Series ſit

$$1 + \frac{2}{x} + \frac{4}{x^2} + \frac{8}{x^3} + \frac{16}{x^4} + \frac{32}{x^5} + \&c.$$

cujus Æquatio eſt $2T - xT' = 0$, erit $r = 2$, $s = -x$; quibus ſcriptis, prodibit $S = \frac{-x}{2-x}T$, vel $S = \frac{x}{x-2}T$. Subſtituatur jam quilibet Terminus pro T, & $\frac{x}{x-2}T$ erit Summa ejus & omnium ſequentium uſque in infinitum. Sit T æqualis primo Termino nempe unitati, & habebitur $\frac{x}{x-2}$ pro valore totius Seriei.

Exemplum II.

Ad eundem modum ſi Æquatio ad tres Terminos ſit $rT + sT' + tT'' = 0$, relatio Summarum erit $rS + sS' + tS'' = 0$; in quâ ſcribendo

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do S—T pro S', & S—T—T' pro S'', emerget $rS+sS-sT+tS-tT-tT'=0$, & exinde $S=\frac{\overline{s+t}T+tT'}{r+s+t}$: adeoque datur S ex datis duobus Terminis. Sit Series ſummanda

$$1+3x+8x^2+21x^3+55x^4+144x^5+\&c.$$

in quâ relatio Terminorum eſt $xxT-3xT'+T''=0$, hinc eſt $r=xx$, $s=-3x$, $t=1$; quibus ſcriptis, habebitur $S=\frac{\overline{1-3x}T+T'}{1-3x+xx}$. Subſtitue jam primum Terminum pro T, & ſecundum pro T', & prodibit $\frac{1}{1-3x+xx}$ pro valore Seriei.

Similiter ſi Æquatio ad Terminos ſit $rT+sT'+tT''+vT'''=0$, erit $S=\frac{\overline{s+t+v}T+\overline{t+v}T'+vT''}{r+s+t+v}$. Et ſic deinceps quando relatio eſt inter plures Terminos.

Scholion.

Notandum relationem Terminorum quæ variabilis eſt, eo propius accedere ad invariabilem, quo magis Termini recedunt ab initio, & tandem in diſtantia infinita evadere omnino invariabilem ut in Seriebus ex Diviſione ortis. Atque hanc appello *ultimam relationem* Terminorum ad quam relatio eorum continue approximat, nunquam tamen perveniet accurate priuſquam Termini remoti fuerint infinito intervallo a principio.

Æquatio autem Differentialis definiens Seriem, rejiciendo dignitates Abſciſſæ omnes præter altiſſimam, & per hanc dividendo Æquationem reſiduam, dabit ultimam relationem Terminorum. Proponatur Æquatio $xxT\times\overline{zz+3z}=T'\times\overline{zz-5z+1}$, rejice omnes dignitates Abſciſſæ infra quadratum, & manebit $xxTzz=T'zz$, quæ diviſa per zz, dabit $xxT=T'$. Et hæc eſt relatio ultima Terminorum.

Ultima autem relatio, quum conſtans ſit, ſuppeditat methodum ſummandi Series quamproxime, in quibus relatio Terminorum eſt variabilis. Si detur Æquatio quævis $rT\times\overline{zz+az+b}+sT'\times\overline{zz+cz+d}=0$, ultima relatio Terminorum erit $rT+sT'=0$, unde $S=\frac{s}{r+s}T$ quamproxime. Hæc Æquatio obtinet accurate quando Terminus T infinite diſtat ab initio, & proxime quando diſtantia eſt notabiliter magna. Similiter ſi Æquatio ſit $rT\times\overline{z+a}+sT'\times\overline{z+b}+tT''\times\overline{z+c}$,

ultima

ultima relatio erit $rT + sT' + tT'' = 0$, & $S = \frac{\overline{s+t}\,T + tT'}{r+s+t}$ proxime. Adeoque colligendo aliquot Terminos initiales, priuſquam inchoatur computus, Summa reliquorum habebitur proxime per hanc methodum. Ex iiſdem principiis etiam corrigere licet approximationem continue ut in Propoſitione ſequente.

Propositio XIV.

Omnis Series $A+B+C+D+E+$ *&c. in quâ ultima relatio Terminorum eſt* $rT + sT' + tT'' = 0$, *aſſumendo* $n = r+s+t$, *& ponendo*

$$\begin{array}{lll} A2 = rA + sB + tC, & A3 = rA2 + sB2 + tC2, & A4 = rA3 + sB3 + tC3, \\ B2 = rB + sC + tD, & B3 = rB2 + sC2 + tD2, & B4 = rB3 + sC3 + tD3, \\ C2 = rC + sD + tE, & C3 = rC2 + sD2 + tE2, & C4 = rC3 + sD3 + tE3, \text{ \&c.} \\ D2 = rD + sE + tF, & D3 = rD2 + sE2 + tF2, & D4 = rD3 + sE3 + tF3, \\ E2 = rE + sF + tG, & E3 = rE2 + sF2 + tG2, & E4 = rE3 + sF3 + tG3, \\ \text{\&c.} & \text{\&c.} & \text{\&c.} \end{array}$$

bipartitur in ſequentes

$$\overline{s+t} \text{ in } \frac{A}{n} + \frac{A^2}{n^2} + \frac{A^3}{n^3} + \frac{A^4}{n^4} + \frac{A^5}{n^5} + \text{\&c.}$$

$$+ t \text{ in } \frac{B}{n} + \frac{B2}{n^2} + \frac{B3}{n^3} + \frac{B4}{n^4} + \frac{B5}{n^5} + \text{\&c.}$$

Per hanc itaque Propoſitionem, Series in quâ ultima relatio Terminorum definitur Æquatione involvente tres Terminos, tranſmutabitur in duas hic exhibitas: quarum tamen prior evaneſcit ubi eſt $s+t=0$. Si relatio ſit inter duos tantum Terminos $rT + sT' = 0$, erit $t = 0$; eaque de cauſa, evaneſcet Series poſterior, & ea quæ tranſmutanda proponitur, migrabit in unicam; quo in caſu particulari coincidet hæc Propoſitio cum Theoremate a D. *Monmort* invento. Si relatio ſit inter plures Terminos quam in hac Propoſitione, Series tranſmutanda migrabit in plures Series. At in omni caſu res erit manifeſta ex eo jam exhibito. Et nolim amittere ſimplicitatem, affectando nimium paucis complecti.

Exemplum

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Exemplum I.

Proponatur Series $1+4x+9x^2+16x^3+25x^4+36x^5+$ &c. Ubi coefficientes ſunt quadrata numerorum naturalium: Æquatio differentialis eſt $x\,T \times \overline{zz+2z+1} - T'zz = 0$, & exinde ultima relatio $xT - T' = 0$, adeoque $r = x$, $s = -1$, $t = 0$, $n = x - 1$; atque

$A = 1$,	$A2 = -3x$,	$A3 = 2x^2$,	$A4 = 0$,
$B = 4x$,	$B2 = -5x^2$,	$B3 = 2x^2$,	$B4 = 0$,
$C = 9x^2$,	$C2 = -7x^3$,	$C3 = 2x^2$,	&c.
$D = 16x^3$,	$D2 = -9x^4$,	&c.	
$E = 25x^4$,	&c.		
&c.			

Omnes igitur Termini poſt A3, evaneſcunt; & ſubſtituendo hoſce valores, Series tranſmutata fit

$$-1 \text{ in } \frac{1}{x-1} - \frac{3}{\overline{x-1}^2} + \frac{2xx}{\overline{x-1}^3}.$$

Et hi tres Termini in unum collecti dant $\frac{1+x}{\overline{1-x}^3}$ pro valore Seriei. Quod ſi mutetur ſignum ipſius x tam in Serie, quam in ejus valore, habebitur

$$\frac{1-x}{\overline{1+x}^3} = 1 - 4x + 9x^2 - 16x^3 + 25x^4 - 36x^5 + \&c.$$

Et inde etiam

$$\frac{1+6x^2+x^4}{\overline{1-xx}^3} = 1 + 9x^2 + 25x^4 + 49x^6 + 81x^8 + \&c.$$

id quod colligi poteſt directe ex Propoſitione abſque hiſce ambagibus.

Exemplum II.

Proponatur Series $1+8x+27x^2+64x^3+125x^4+216x^5+$ &c. ubi Coefficientes ſunt cubi numerorum naturalium: & Æquatio ad Seriem eſt $xT \times \overline{z^3+3z^2+3z+1} - T'z^3 = 0$; adeoque ultima relatio $xT - T' = 0$; & inde $r = x$, $s = -1$, $t = 0$, $n = x - 1$; &

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$A=1$,	$A2=-7x$,	$A3=12x^2$,	$A4=-6x^3$,	$A5=0$,
$B=8x$,	$B2=-19x^2$,	$B3=18x^3$,	$B4=-6x^4$,	$B5=0$,
$C=27x^2$,	$C2=-37x^3$,	$C3=24x^4$,	$C4=-6x^5$,	&c.
$D=64x^3$,	$D2=-61x^4$,	$D3=30x^5$,	&c.	
$E=125x^4$,	$E2=-91x^5$,	&c.		
$F=216x^5$,	&c.			
&c.				

Hic evaneſcunt Termini poſt A4; & valores ſubſtituti in Theoremate, dant

$$-1 \text{ in } \frac{1}{x-1} - \frac{7x}{\overline{x-1}^2} + \frac{12x^2}{\overline{x-1}^3} - \frac{6x^3}{\overline{x-1}^4},$$

& hi quatuor Termini collecti in unum dant $\frac{1+4x+x^2}{\overline{1-x}^4}$ pro valore Seriei.

Exemplum III.

Detur Series ſummanda $1-6x+27x^2-104x^3+366x^4-1212x^5+3842x^6-11784x^7+$&c. quæ definitur Æquatione $xxT\overline{z+4}-2xT'\overline{z+2}-T''z=0$, evadat jam z infinite magna, & ultima relatio erit $xxT-2xT'-T''=0$: hinc fit $r=xx$, $s=-2x$, $t=-1$, $n=xx-2x-1$; atque

$A=+1$,	$A2=-14x^2$,	$A3=+29x^4$,	$A4=0$,
$B=-6x$,	$B2=+44x^3$,	$B3=-70x^5$,	$B4=0$,
$C=+27x^2$,	$C2=-131x^4$,	$C3=+169x^6$,	&c.
$D=-104x^4$,	$D2=+376x^5$,	$D3=-408x^7$,	
$E=+366x^4$,	$E2=-1052x^6$,	&c.	
$F=-1212x^5$,	$F2=+2888x^7$,		
$G=+3842x^6$,	&c.		
$H=-11784x^7$			
&c.			

Subſtitue jam hoſce valores pro A, A2, A3, B, B2, B3, atque Series propoſita tranſmigrabit in ſequentes duas numero Terminorum finitas,

$$-\overline{2x+1} \text{ in } \frac{1}{xx-2x-1} - \frac{14x^2}{\overline{xx-2x-1}^2} + \frac{29x^4}{\overline{xx-2x-1}^3},$$

$$\frac{-6x}{xx-2x-1} + \frac{44x^3}{\overline{xx-2x-1}^2} - \frac{70x^5}{\overline{xx-2x-1}^3}.$$

Quæ duæ in unam Summam collectæ dant $\frac{1}{\overline{1+2x-xx}^2}$ pro valore Seriei. Ad eundem modum Theorema applicatur ad Tranſmutationem Serierum quæ ſummari nequeunt. Demonſtratio autem ex ſequentibus fiet

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fiet manifeſta. Sit Fractio quæcunque $\frac{a+bx+cx^2+dx^3}{\overline{v+tx+sx^2+rx^3}|^n}$, cujus Numerator ſit quantitas conſtans ex membris quotcunque, quorum tamen numerus eſt determinatus. Sit vero ejus Denominator dignitas quælibet quantitatis, quæ etiam conflatur ex membris quotvis Numero finitis. Tum quicunque ſit index n, ſi Fractio reſolvatur in Seriem, eadem ſemper erit ultima relatio Terminorum, ac ſi Denominator fuiſſet ſimplex poteſtas $v+tx+sx^2+rx^3$. In hanc igitur ſi Series continue multiplicetur, factum tandem abrumpet, ſi ſit n integer & affirmativus: hoc eſt, ſi Series ſummabilis ſit per ſimplicem Æquationem.

Proponatur Series $1-6x+27x^2-104x^3+366x^4-$ &c. in quâ ultima relatio Coefficientium eſt $A-2B-C=0$; ſumo quantitatem $xx-2x-1$, vel mutatis ſignis, $1+2x-xx$, in qua Coefficientes iidem ſunt ac in ultima relatione, & concludo Seriem propoſitam, modo ſummabilis ſit, æqualem eſſe alicui Fractioni cujus Denominator eſt Dignitas quædam quantitatis $1+2x-xx$. Ergo pono $S=1-6x+27x^2-104x^3+366x^4-$ &c. & duco utramque partem in $1+2x-xx$; atque prodit $S\times\overline{1+2x-xx}=1-4x+14x^2-44x^3+131x^4-$ &c. quam quoniam nondum abrumpit, iterum duco in eandem quantitatem, & habeo $S\times\overline{1+2x-xx}|^2=1-2x+5x^2-12x^3+29x^4-$ &c. dein tertia vice multiplicans eandem, obtineo $S\times\overline{1+2x-xx}|^3=1$; Terminis omnibus poſt primum evaneſcentibus.

Igitur palam eſt hanc Propoſitionem nihil aliud præſtare quam methodum compendioſam multiplicationis, & ſimul diſpoſitionem Terminorum in Seriebus tranſmutatis quæ efficit celerem convergentiam quando non abrumpunt. Quare Demonſtrationem Lectori indagandam relinquo, quæ quidem eadem fere eſt ſive Denominatores ſint trinomiales, quadrinomiales aut plurium nominum.

Propositio XV.

Invenire Æquationem ſive Algebraica ſive Fluxionalis ſit, cujus Radix erit Series quæcunque data quæ definitur Æquatione in qua Termini Seriei ſunt unius tantum dimenſionis.

Series datur ex datis aliquot Terminis initialibus & ſimul lege formandi reliquos: ex hiſce autem datis invenietur Æquatio habens Seriem illam pro Radice, id quod per Exempla ſequentia intelligetur.

Exemplum

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Exemplum I.

Invenire Æquationem cujus Radix eſt Series $A+Bx+Cx^2+Dx^3+Ex^4+$ &c. in quâ relatio Coefficientium eſt conſtans, ſcilicet $rA+sB+tC+vD=0$. Quoniam relatio eſt invariabilis, concludo Seriem æquari Fractioni rationali, quam ſic eruo. Fingo y æqualem Serie datæ: & ſumendo indices relationis r, s, t, v, in ordine inverſo, duco y ejuſque valorem ſucceſſive in v, tx, sx^2, rx^3, atque provenient

$$\begin{aligned} vy &= vA+vBx+vCx^2+vDx^3+vEx^4+vFx^5+\text{\&c.} \\ txy &= \quad tAx+tBx^2+tCx^3+tDx^4+tEx^5+\text{\&c.} \\ sx^2y &= \quad\quad sAx^2+sBx^3+sCx^4+sDx^5+\text{\&c.} \\ rx^3y &= \quad\quad\quad rAx^3+rBx^4+rCx^5+\text{\&c.} \end{aligned}$$

Sed ex ſuppoſitione, eſt $rA+sB+tC+vD=0$, $rB+sC+tD+vE=0$, & ſic in infinitum. Unde evaneſcunt omnes Termini in quibus x eſt plurium quam duarum dimenſionum; adeoque reſtabit

$$vy+txy+sx^2y+rx^3y=vA\left.\begin{matrix}+vB\\+tA\end{matrix}\right\}x\left.\begin{matrix}+vC\\+tB\\+rA\end{matrix}\right\}x^2.$$

Quæ eſt Æquatio determinans valorem Seriei. Exempli cauſâ, ſit $A=2$, $B=-3$, $C=7$, $r=3$, $s=-7$, $t=6$, $v=4$; & evadet Æquatio

$$4y+6xy-7x^2y+3x^3y=8x-4x^2, \text{ vel } y=\frac{8x-4x^2}{4+6x-7x^2+3x^3}.$$

Exemplum II.

Invenire Æquationem quæ pro Radice habeat Seriem $1-\frac{2}{3}x^2+\frac{8}{15}x^4-\frac{16}{35}x^6+\frac{128}{315}x^8-\frac{256}{693}x^{10}+$ &c. in qua relationes Coefficientium ſunt $2A+3B=0$, $4B+5C=0$, $6C+7D=0$, $8D+9E=0$, &c. In hoc Exemplo Æquatio quæſita involvet Fluxiones primas, quoniam numeri in relationibus ſunt æqui differentes. Aſſume ergo

$$\begin{aligned} y &= A+Bx^2+Cx^4+Dx^6+Ex^8+\text{\&c.} \\ \text{erit } x\dot{y} &= \quad 2Bx^2+4Cx^4+6Dx^6+8Ex^8+\text{\&c.} \\ y+x\dot{y} &= A+3Bx^2+5Cx^4+7Dx^6+9Ex^8+\text{\&c.} \\ 2x^2y+x^3\dot{y} &= 2Ax^2+4Bx^4+6Cx^6+8Dx^8+\text{\&c.} \end{aligned}$$

Adjiciatur nunc ultima Æquatio penultimæ atque prodibit

$y+$

$$y+x\dot{y}+2x^2y+x^3\dot{y}=A\left.\begin{matrix}+2A\\+3B\end{matrix}\right\}x^2\left.\begin{matrix}+4B\\+5C\end{matrix}\right\}x^4\left.\begin{matrix}+6C\\+7D\end{matrix}\right\}x^6\left.\begin{matrix}+8D\\+9E\end{matrix}\right\}x^8+\&c.$$

Sed ex hypotheſi relationes Coefficientium ſunt $2A+3B=0$, $4B+5C=0$, $6C+7D=0$, &c. ideoque Termini omnes poſt primum A evaneſcunt, & reſtat Æquatio finita $y+x\dot{y}+2x^2y+x^3\dot{y}=A=1$, ſive $y\times\overline{1+2xx}+x\dot{y}\times\overline{1+xx}=1$: ſubſtituta ſcilicet unitate, ſive primo Termino Seriei pro Coefficiente A.

Exemplum III.

Invenienda ſit Æquatio cujus Radix eſt Series $1-\frac{1}{4}x^2-\frac{3}{64}x^4-\frac{5}{256}x^6-\frac{175}{2048}x^8-$&c. ubi relationes Coefficientium ſunt

$-1.1A-2.2B=0$, $1.3B-4.4C=0$, $3.5C-6.6D=0$, $5.7D-8.8E=0$, &c.

Vel $-A-4B=0$, $3B-16C=0$, $15C-36D=0$, $35D-64F=0$, &c.

In hoc caſu Æquatio deſiderata neceſſario involvet Fluxiones ſecundas, quoniam numeri qui indicant relationem Coefficientium, ſunt duarum dimenſionum, ſive facta ſub numeris æquidifferentibus binatim ſumptis. Sit itaque

$$y=A+Bx^2+Cx^4+Dx^6+Ex^8+\&c.$$
$$\dot{y}=2Bx+4Cx^3+6Dx^5+8Ex^7+\&c.$$
$$x\ddot{y}=2Bx+12Cx^3+30Dx^5+56Ex^7+\&c.$$
$$\dot{y}+x\ddot{y}=4Bx+16Cx^3+36Dx^5+64Ex^7+\&c.$$
$$x^3\ddot{y}+x^2\dot{y}-xy=-Ax+3Bx^3+15Cx^5+35Dx^7+\&c.$$

Denique ſubducendo hanc ultimam Æquationem de penultima, manebit

$$x\ddot{y}-x^3\ddot{y}+\dot{y}-x^2\dot{y}+xy=\left\{\begin{matrix}+A\\+4B\end{matrix}\right\}x\left.\begin{matrix}-3B\\+16C\end{matrix}\right\}x^3\left.\begin{matrix}-15C\\+36D\end{matrix}\right\}x^5\left.\begin{matrix}-35D\\+64E\end{matrix}\right\}x^7\&c.$$

hoc eſt $x\ddot{y}-x^3\ddot{y}+\dot{y}-x^2\dot{y}+xy=0$, vel $\dot{y}+x\ddot{y}+\frac{xy}{1-xx}=0$. Nam propter relationem Coefficientium, omnia membra ex una parte Æquationis evaneſcunt.

Exemplum IV.

Quæratur Æquatio, cujus Radix eſt $1+\frac{1}{4}Ax^2+\frac{9}{10}Bx^4+\frac{25}{36}Cx^6$

 $+\frac{49}{64}$

$+\frac{49}{64}Dx^{8}+$ &c. ubi relationes Coefficientium ſunt $A-4B=0$, $9B-16C=0$, $25C-36D=0$, &c. Finge

$$y=A+Bx^{2}+Cx^{4}+Dx^{6}+Ex^{8}+Fx^{10}+\&c.$$

$$\text{Erit } \dot{y}=\quad 2Bx+4Cx^{3}+6Dx^{5}+8Ex^{7}+10Fx^{9}+\&c.$$

$$\text{ac } x\ddot{y}=\quad 2Bx+12Cx^{3}+30Dx^{5}+56Ex^{7}+90Fx^{9}+\&c.$$

Dein per computum invenies

$$xy+3x^{2}\dot{y}+x^{3}\ddot{y}=Ax+9Bx^{3}+25Cx^{5}+49Dx^{7}+81Ex^{9}+\&c.$$

$$\dot{y}+x\ddot{y}=4Bx+16Cx^{3}+36Dx^{5}+64Ex^{7}+100Fx^{9}+\&c.$$

Harum Æquationum poſteriorem ſubducito de priore & habebis

$$xy-\dot{y}\times\overline{1-3xx}-x\ddot{y}\times\overline{1-xx}=\left\{\begin{matrix}+A\\-4B\end{matrix}\right\}x\left\{\begin{matrix}+9B\\-16C\end{matrix}\right\}x^{3}\left\{\begin{matrix}+25C\\-36D\end{matrix}\right\}x^{5}\left\{\begin{matrix}+49D\\-64E\end{matrix}\right\}x^{7}$$
$$+\&c.$$

Et inde propter relationem Coefficientium, Æquatio erit $xy=\dot{y}\times\overline{1-3xx}+x\ddot{y}\times\overline{1-xx}$.

Exemplum V.

Quæratur Æquatio cujus Radix eſt $x+\frac{1}{4}x^{2}+\frac{1}{9}x^{3}+\frac{1}{16}x^{4}+$ &c. ubi Denominatores ſunt quadrata numerorum 1, 2, 3, &c. Aſſume

$$y=x+\frac{1}{4}x^{2}+\frac{1}{9}x^{3}+\frac{1}{16}x^{4}+\frac{1}{25}x^{5}+\&c.$$

$$\text{Erit } x\dot{y}=x+\frac{1}{2}x^{2}+\frac{1}{3}x^{3}+\frac{1}{4}x^{4}+\frac{1}{5}x^{5}+\&c.$$

Cujus Fluxio eſt

$$x\ddot{y}+\dot{y}=1+x+x^{2}+x^{3}+x^{4}+\&c.$$

$$\text{hoc eſt } x\ddot{y}+\dot{y}=\frac{1}{1-x}.$$

Similiter ſi Series ſit $x+\frac{1}{8}x^{2}+\frac{1}{27}x^{3}+\frac{1}{64}x^{4}+$ &c. ubi Denominatores ſunt Cubi, invenies Æquationem eſſe $\dot{y}+3x\ddot{y}+x^{2}\dddot{y}=\frac{1}{1-x}$. Et ſi Series ſit $x+\frac{1}{16}x^{2}+\frac{1}{81}x^{3}+\frac{1}{256}x^{4}$ &c. exiſtentibus Denominatoribus

toribus biquadratis, prodibit Æquatio $\dot{y} + 6x\ddot{y} + 7x^2\dddot{y} + x^3\ddddot{y} = \frac{1}{1-x}$. Et ſic deinceps.

Scholion.

Ex data igitur relatione Coefficientium, Series reducuntur vel ad Fractiones vel ad Fluxiones, idque eâdem prorſus facilitate in omnibus caſibus; ſive Series determinetur per relationem plurium vel paucio-rum Terminorum. Nam quoniam ſumendo Fluxionem Seriei, Termini ducuntur in indices dignitatum, qui ſemper ſunt æquidifferentes; & rurſus in alias quantitates æquidifferentes, ſumendo Fluxiones ſecundas; & ſic porro, formare licet hac ratione relationes Terminorum quæcunque ſint, inſiſtendo veſtigiis mox traditis.

Ad ſummationem autem perfectam oportet Fluxiones reduci ad Fluentes; quod ubi fieri nequit, concludendum eſt, Seriem de qua agitur non eſſe ex ipſarum numero quæ ſummari poſſunt. Multum autem confert ad promovendam hanc doctrinam ut Summationes reducantur ad Fluxiones, quoniam methodus regrediendi a fluxionibus ad Fluentes, licet imperfecta ſit, eſt tamen melius nota & magis exculta quam ea regrediendi a Differentiis ad Summas. Sed & altera alteri mutuo præbebit opem: Fluentes enim ſi inveniri poſſint, exhibebunt Summas; vel e contra Summæ inventæ dabunt Fluentes.

Ex Propoſitionibus jam traditis, aliiſque ex iiſdem principiis facile deducendis, invenire licet Radicem Æquationis cujuſcunque accuratiſſime in numeris modo reduci poſſit in Seriem ſimplicem, etiamſi lentiſſime convergentem. Sed etiamnum reſtat difficultas, ſi neque Series cito approximet neque ſit relatio Terminorum ſimplex & idonea quæ definiatur Æquatione differentiali. Quippe ſi in Æquatione reſolvenda reperiatur Radix ejuſve Fluxiones plurium quam unius dimenſionis, vel ſi hæ in ſe mutuo multiplicentur; & hujuſmodi membra nequeant eliminari: in iis inquam caſibus Radix non eſt reducibilis in Seriem ſimplicem compoſitam ex dignitatibus Abſciſſæ, ſaltem per artes mihi hactenus notas. At dignitates Abſciſſæ utcunque altæ ſint in Æquatione reſolvendâ, nullatenus obſtant ſimplicitati Seriei.

Sed hac oblata occaſione, conabor tollere difficultatem hactenus intactam, quæ tamen in reductione Æquationum Fluxionalium maxima eſt. Sciendum igitur eſt eas Æquationes Fluxionales quæ involvunt omnes conditiones Problematis, ideoque determinant omnes Coefficientes in ſuis Radicibus, reſolvi poſſe perinde ac Æquationes literales affectæ, idque per regulas a *Newtono* dudum traditas. Attamen plerumque fit Æquationes non determinare omnes Coefficientes Terminorum

rum in ſuis Radicibus; ſcilicet quoniam quantitates conſtantes evaneſcunt & omnino amittuntur ex Æquatione in progreſſu a Fluentibus ad Fluxiones. Sed & variis modis evaneſcere ſolent prout Fluxio ſumitur, id quod Exemplo declarare libet.

Proponatur Æquatio $y^2 = a^2 + bx + \frac{x^4}{c^2}$; & fluente x uniformiter, atque ſcripta unitate pro ejus Fluxione; habebitur per Methodum directam $2y\dot{y} = b + \frac{4x^3}{c^2}$: cujus Radix per methodos *Newtoni* extracta, eadem erit ac Radix Fluentis $y^2 = bx + \frac{x^4}{c^2}$, idque propter quantitatem aa jam amiſſam. Et hoc in caſu non determinatur primus Terminus Seriei, qui utique pendet ex evaneſcente quantitate aa.

Similiter ſi Æquatio propoſita prius dividatur per x, exibit $\frac{y^2}{x} = \frac{aa}{x} + b + \frac{x^3}{c^2}$, cujus Fluxio eſt $\frac{2y\dot{y}}{x} - \frac{y^2}{x^2} = \frac{-aa}{x^2} + \frac{3x^2}{cc}$, vel ducendo in x^2, $2xy\dot{y} - y^2 = -aa + \frac{3x^4}{cc}$. Et hujus Radix pure ſumpta eadem eſt ac Fluentis, $y^2 = a^2 + \frac{x^4}{cc}$. Quo in caſu non determinatur ſecundus Terminus Seriei, qui dependet ex Coefficiente b nunc amiſſâ.

Denique ſi Æquatio fluens dividatur per x^4, habebitur $\frac{y^2}{x^4} = \frac{aa}{x^4} + \frac{b}{x^3} + \frac{1}{cc}$, & exinde Fluxio erit $\frac{2y\dot{y}}{x^4} - \frac{4y^2}{x^5} = \frac{-4aa}{x^5} - \frac{3b}{x^4}$, vel $2xy\dot{y} - 4y^2 = -4aa - 3bx$; quæ reſoluta per regulas vulgares, eandem dabit Radicem atque Fluens $y^2 = a^2 + bx$: adeoque Terminus Seriei quintus, qui dependet ex Coefficiente cc non determinatur. Hinc patet fluxionales Æquationes

$$2y\dot{y} = b + \frac{4x^3}{c^2},$$

$$2xy\dot{y} - y^2 = -aa + \frac{3x^4}{cc},$$

$$2xy\dot{y} - 4y^2 = -4aa - 3bx$$

provenire ab unicâ fluente $y^2 = aa + bx + \frac{x^4}{cc}$: per primam Terminus primus Seriei non determinatur; per ſecundam determinatur primus at

non

non ſecundus Terminus; per tertiam denique determinantur primi quatuor Termini Seriei, at quintus relinquitur indeterminatus. Poſſunt igitur diverſæ Æquationes fluxionales provenire ex eâdem fluente. Sed methodus reſolutionis non erit perfecta priuſquam in poteſtate fuerit exhibere omnes Radices diverſarum fluentium a quibus Fluxio quævis propoſita ullo modo pervenire poſſit. Nam neceſſe eſt habere rationem quantitatum quæ vel actu evanuere vel evaneſcere potuerunt.

Etenim non licet extrahere Radicem ex Æquatione fluxionali tanquam nulla quantitas evanuiſſet, dein addere quantitatem datam fluenti inventæ ut in Quadratura Curvarum. Termini utique indeterminati ſæpiſſime ſunt quantitates invariabiles: & additio Quantitatis datæ Radici inventæ non æquipollet additioni Quantitatis Æquationi reſolvendæ. Sic eadem Fluxio $2y\dot{y} = b + \frac{4x^3}{cc}$ provenire poteſt ab utrâvis fluentium $yy = aa + bx + \frac{x^4}{cc}$, vel $yy = bx + \frac{x^4}{cc}$; quarum tamen Radices ſortiuntur formas omnino diverſas, ſcilicet prior $A + Bx + Cx^2 +$ &c. poſterior $Ax^{\frac{1}{2}} + Bx^{\frac{3}{2}} + Cx^{\frac{5}{2}} +$ &c.

Inſuper fieri poteſt Æquationem fluxionalem involvere omnes Coefficientes quas involvit fluens, & hoc non obſtante, Terminos omnes non determinari, ut in Exemplo ſequente. Æquatio $y^2 = aa + bx + \frac{x^4}{cc}$ exhibet directe Fluxionem $2y\dot{y} = b + \frac{4x^3}{cc}$: quæ ſi ducatur in y, evadet $2y^2\dot{y} = by + \frac{4x^3y}{cc}$; & dein ſubſtituatur pro yy ſuus valor $aa + bx + \frac{x^4}{cc}$, tandem habebitur $2aa\dot{y} + 2bx\dot{y} + \frac{2x^4\dot{y}}{cc} = by + \frac{4x^3y}{cc}$. Et hæc eſt Æquatio fluxionalis involvens omnes Coefficientes quas involvit ejus fluens; in qua nihilominus primus Terminus Seriei non determinatur. Idem etiam ubique eveniet niſi adſit membrum in Æquatione quod nec involvit Radicem neque ejus Fluxionem quamlibet.

Sunt & aliæ difficultates quamplurimæ haud minoris momenti, quas facile quivis percipiet multiplicari in Fluxionibus inferiorum ordinum: nam in Fluxionibus ſecundis, Termini duo ab invicem neutiquam pendentes poſſunt eſſe indeterminati; & tres in tertiis; & ſic porro. Cæterum accidit haud raro omnes Terminos determinari ex Æquatione etiam involvente Fluxiones inferiorum Ordinum. Et omnia quæ hic dicta ſunt de Fluxione cujus fluens nota eſt, etiam vera ſunt de Fluxionibus quarum fluentes ſunt ignotæ.

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Radix Æquationis cujuſcunque eſt quantitas, quæ ſcripta pro literâ denotante Radicem, efficiet omnes Terminos Æquationis reſultantis evaneſcere. Termini autem duobus tantum modis poſſunt evaneſcere, vel per ſignorum contrarietatem in membris homologis, vel etiam ubi reperitur quantitas conſtans in fluente; eadem enim nullum relinquit veſtigium in Fluxione. Ut ſi ſit $y = Ax^n$; erit $\dot{y} = nAx^{n-1}$, $\ddot{y} = \overline{nn-n}\,Ax^{n-2}$; ſi ſit $n = 0$, evaneſcet valor Fluxionis primæ; ſi ſit $nn-n = 0$, id eſt, $n = 0$, vel $n = 1$, evaneſcet valor ſecundæ; idque abſque aliis membris homologis quæ eas deſtruere poſſint. Et hæc ſunt principia per quæ enodandæ ſunt difficultates univerſæ quæ occurrunt in reſolutione Æquationum fluxionalium.

Proponatur Æquatio reſolvenda $r^2\dot{y}^2 = r^2\dot{x}^2 - \dot{x}^2y^2$, vel poſita unitate pro $\dot{x}$, $r^2\dot{y}^2 = r^2 - y^2$. Per parallelogrammum vel alias *Newtoni* methodos, invenies Radicem

$$y = x - \frac{x^3}{6r^2} + \frac{x^5}{120r^4} - \frac{x^7}{5040r^6} + \&c.$$

Hujus enim quadratum ſcriptum pro y^2, & Fluxionis quadratum pro $\dot{y}^2$ efficiet omnia membra Æquationis reſultantis ſeſe deſtruere per contrarietatem ſignorum. Videamus autem an alia ſit radix quæ hoc modo inveniri nequeat. Illum in finem finge Ax^n eſſe primum Terminum Seriei, ſive $y = Ax^n$ proxime; eritque $\dot{y} = nAx^{n-1}$; adeoque $\dot{y}^2 = n^2A^2x^{2n-2}$, & $y^2 = A^2x^{2n}$; quibus ſcriptis, reſultat $n^2r^2A^2x^{2n-2} = r^2 - A^2x^{2n}$, vel ducendo totam in x^2.

$$n^2r^2A^2x^{2n} = r^2x^2 - A^2x^{2n+2}.$$

Ubi patet membrum $n^2r^2A^2x^{2n}$ evaneſcere quando eſt $n = 0$: ſubſtituatur ergo 0 pro n, & evadet Æquatio

$$0 \times r^2A^2x^0 = r^2x^2 - A^2x^2$$

hoc igitur in caſu evaneſcit membrum $r^2A^2x^0$ in quo x eſt minimarum dimenſionum: eâque de cauſâ erit Ax^0, ſive conſtans quantitas A primus Terminus Seriei eo citius convergentis quo minor eſt x. Atque hunc computum proſequendo invenies per methodos vulgares

$$y = A \times \overline{1 - \frac{x^2}{2r^2} + \frac{x^4}{24r^4} - \frac{x^6}{720r^6} + \&c.}$$

Sed quantitas A ex Æquatione Fluxionali non determinatur. Quod ſi Æquationis $r^2\dot{y}^2 = r^2 - y^2$, capiatur Fluxio habebitur $2r^2\dot{y}\ddot{y} = -2y\dot{y}$, vel dividendo & tranſponendo $r^2\ddot{y} + y = 0$. Ponamus jam eſſe $y = Ax^n$, erit

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erit $\dot{y}=nAx^{n-1}$, & $\ddot{y}=\overline{nn-n}\,Ax^{n-2}$, quibus ſubſtitutis, prodit $\overline{nn-n}\times r^2Ax^{n-2}+Ax^n=0$; ubi patet indicem $n-2$ non comparari poſſe cum indice n; & proinde nullam Radicem illâ methodo erui poſſe. Attamen ponendo Coefficientem $nn-n=0$, habebitur $n=0$, & $n=1$; adeoque poteſt A vel Ax eſſe primus Terminus Seriei, uti prius invenimus.

Prior Series dat Sinum, & poſterior Coſinum ex dato Arcu x. Eſtque Coefficiens A in poſteriore æqualis Radio r. Etenim ſive ponas y pro Sinu vel Coſinu, ſemper incides in Æquationem $r^2\dot{y}^2=r^2-y^2$, quæ nullâ Radice explicabilis eſt præter duas jam exhibitas. Ex hoc autem Exemplo notare licet formam Seriei ſemper determinatam eſſe ex Æquatione, etiamſi Coefficientes non determinentur. Et ubi Coefficiens cujuſvis Termini non determinatur, index ipſius x in eodem ſemper invenietur ponendo membrum aliquod Æquationis reſultantis æquale nihilo. Ubi vero Coefficiens Termini determinatur, index ipſius x in eodem invenitur per comparationem duorum indicum in Æquatione reſultante, idque per regulas *Newtoni*.

Reſolvenda ſit Æquatio $\ddot{y}+a^2y-x\dot{y}-x^2\ddot{y}=0$, ubi x fluit uniformiter, & unitas ſcribitur pro ejus Fluxione. Finge $y=Ax^n$ proxime, eritque $\dot{y}=nAx^{n-1}$, $\ddot{y}=\overline{nn-n}\,Ax^{n-2}$; quibus ſubſtitutis in Æquatione reſultat $\overline{nn-n}\,Ax^{n-2}+\overline{aa-nn}\,Ax^n=0$. Pone jam Coefficientem $nn-n=0$, & prodibit $n=0$, vel $n=1$: hi valores ſubſtituti pro n in Æquatione, exhibent

$$0\times Ax^{-2}+aaAx^0=0,\ \&$$
$$0\times Ax^{-1}+\overline{aa-1}\,Ax=0;$$

quumque in utroque caſu evaneſcat membrum in quo x eſt minimarum dimenſionum, concludo eſſe 0 vel 1, indicem ipſius x in primo Termino Seriei quæ convergit eo citius quo minor eſt x: adeoque uſurpare licet

$$y=A+Bx^2+Cx^4+Dx^6+\&c.$$
$$\text{vel } y=Ax+Bx^3+Cx^5+Dx^7+\&c.$$

In Æquatione reſultante $\overline{nn-n}\,Ax^{n-2}+\overline{aa-nn}\,Ax^n=0$, pone nunc Coefficientem $aa-nn=0$; eritque $n=\pm a$: ſcribe igitur $+a$, & $-a$ pro n, & prodibit

$$\overline{aa-a}\,Ax^{a-2}+0\times Ax^a=0$$
$$\overline{aa+a}\,Ax^{-a-2}+0\times Ax^{-a}=0.$$

In utroque igitur caſu evaneſcit membrum in quo x eſt maximarum dimenſionum; eaque propter eritque $+a$ vel $-a$ index primi Termini Seriei eo citius convergentis quo major eſt x. Nam aſſumere licet

$y=$

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$$y = Ax^a + Bx^{a-2} + Cx^{a-4} + Dx^{a-6} + \&c.$$

$$\text{vel } y = Ax^{-a} + Bx^{-a-2} + Cx^{-a-4} + Dx^{-a-6} + \&c.$$

Harum quatuor Serierum coefficientes determinatæ dant valores Radicis ſequentes

$$A + \frac{0-aa}{1.2} Ax^2 + \frac{4-aa}{3.4} Bx^2 + \frac{16-aa}{5.6} Cx^2 + \frac{36-aa}{7.8} Dx^2 + \&c.$$

$$Ax + \frac{1-aa}{2.3} Ax^2 + \frac{9-aa}{4.5} Bx^2 + \frac{25-aa}{6.7} Cx^2 + \frac{49-aa}{8.9} Dx^2 + \&c.$$

$$Ax^a - \frac{a.\overline{a-1}}{4.\overline{a-1}} \times \frac{A}{x^2} - \frac{\overline{a-2}.\overline{a-3}}{8.\overline{a-2}} \times \frac{B}{x^2} - \frac{\overline{a-4}.\overline{a-5}}{12.\overline{a-3}} \times \frac{C}{x^2} - \frac{\overline{a-6}.\overline{a-7}}{16.\overline{a-4}} \times \frac{D}{x^2} - \&c.$$

$$\frac{A}{x^a} - \frac{a.\overline{a+1}}{4.\overline{a+1}} \times \frac{A}{x^2} - \frac{\overline{a+2}.\overline{a+3}}{8.\overline{a+2}} \times \frac{B}{x^2} - \frac{\overline{a+4}.\overline{a+5}}{12.\overline{a+3}} \times \frac{C}{x^2} - \frac{\overline{a+6}.\overline{a+7}}{16.\overline{a+4}} \times \frac{D}{x^2} - \&c.$$

Duæ priores attinent ad multiplicationem aut diviſionem Arcus circularis. Poſteriores vero ſpectant ad aream Hyperbolæ. Et hæc Exempla ſufficiant ad illuſtrandam regulam hic adductam pro determinando indice. Ea deducitur ex Propoſitione quintâ: atque per eandem & regulas *Newtoni*, extractio radicum in Seriebus infinitis ad umbilicum perducitur, uti quivis facile percipiet qui mediocriter verſatus eſt in iis quæ hâc de re Authores hactenus protulere. Etenim facile demonſtrabitur Æquationem non admittere Seriem pro Radice quæ hâc methodo non eruitur. Sed hic non loquor de Seriebus quæ conflantur ex Terminis in quibus indeterminata x habet indices indeterminatos.

PARS

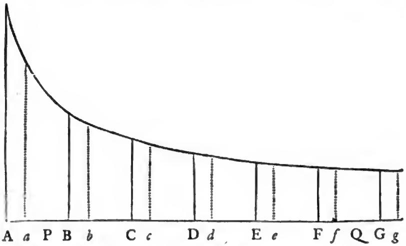
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PARS SECUNDA.

DE

Interpolatione Serierum.

SIT recta quævis poſitione data PQ, ſuper quam erigantur Ordinatæ quotcunque A, B, C, D, &c. ſibi mutuo parallelæ, & æqualibus intervallis ab invicem diſtantes: deſignent autem hæ Ordinatæ Terminos Seriei regularis, continue increſcentes vel decreſcentes, & eodem ſigno affectos; atque una eademque Curva tranſibit per extremitates omnium, quæ quidem determinabitur ex data



Æquatione ad Seriem, id eſt, ex datâ Æquatione generaliter exprimente relationem inter duas vel plures Ordinatas quaſvis ſucceſſivas.

Si ex data Æquatione illa differentiali erui poſſit Æquatio hujus Curvæ Algebraica, utpote quæ definit relationem inter Abſciſſas & Ordinatas correſpondentes, habebitur Ordinata quæcunque ex data ejus Abſciſſâ per reſolutionem Æquationum affectarum, adeoque Interpolatio Seriei abſoluta; quæ utique conſiſtit in aſſignatione cujuſlibet Termini primarii vel intermedii ex dato ejus loco in Serie. Ubi vero Æquatio Curvæ Algebraica nequit inveniri, id quod plerumque fit, nihil ulterius ſperandum eſt quam exhibere valorem Termini cujuſlibet quæſiti per Seriem convergentem, vel forte per Quadraturam Curvarum.

Hic autem loquor de Terminis æquidiſtantibus, quorum quippe relationes prodeunt ſcribendo numeros æquidifferentes ſucceſſive pro Abſciſſâ z in Æquatione differentiali Seriem definiente. Nam intervallum commune Ordinatarum ſuper Abſciſſam conſiſtentium, proportionale eſt incremento invariabili indeterminatæ z.

Propositio XVI.

Si idem ſit intervallum commune Terminorum primariorum & intermediorum, & detur unus ex intermediis, dabuntur omnes ex data Æquatione ad primarios.

In figurâ ſuperiore deſignent A, B, C, D, &c. Terminos primarios, & *a*, *b*, *c*, *d*, &c. intermedios; ſintque intervalla primariorum AB, BC, CD, &c. æqualia intervallis intermediorum *ab*, *bc*, *cd*, &c. quodque cuique. Dico omnes intermedios dari ex dato eorum quovis uno & ſimul Æquatione differentiali definiente relationem primariorum.

Etenim Æquatio quæ aſſignat relationem primariorum definit Curvam tranſeuntem per eorum extremitates; & Æquatio quæ definit relationem intermediorum definit etiam Curvam tranſeuntem per illorum extremitates. Sed per definitionem primariorum & intermediorum eadem Curva tranſit per amborum extremitates; quare eadem Æquatio quæ definit Curvam, definiet relationem Terminorum in utraque Serie. Et illa Æquatio, datur ex hypotheſi; adeoque datur lex continuandi intermedios, qui itaque omnes dabuntur ex dato quovis uno. Q. E. D.

Corollarium. Si in Æquatione quâvis differentiali exprimente relationem primariorum A, B, C, D, &c. valores ſucceſſivi Abſciſſæ z ſint —PA, PB, PC, PD, &c. exiſtente puncto quovis P principio Abſciſſæ: tum

tum habebuntur relationes intermediorum ſcribendo pro z ſucceſſive $-Pa$, Pb, Pc, Pd, &c. in eâdem Æquatione. Nam Æquatio differentialis generaliter exprimit relationem inter duas quaſcunque Ordinatas ad certam diſtantiam ab invicem ſitas, ſive eæ cadant in Serie primariarum ſive in eâ intermediarum.

Exemplum I.

In progreſſione Terminorum Geometrica $1, x, x^2, x^3, x^4$, &c. ſi Termini in medio conſiſtentes inter primarios ſint a, b, c, d, e, &c. erit $b = ax$, $c = bx$, $d = cx$, $e = dx$, &c. quorum utique eadem eſt relatio ac primariorum.

Exemplum II.

Si Termini primarii ſint 1, 1, 2, 6, 24, 120, 720, &c. quorum relationes ſunt $B = A$, $C = 2B$, $D = 3C$, $E = 4D$, &c. & Terminus directe in medio conſiſtens inter duos primos 1 & 1 ſit a; reliqui dabuntur per Æquationes $b = \frac{3}{2}a$, $c = \frac{5}{2}b$, $d = \frac{7}{2}c$, $e = \frac{9}{2}d$, &c. hi utpote conſiſtunt in medio inter duos quoſque primarios, ab iiſdem hinc inde æqualiter diſtantes.

Si vero a deſignet Terminum inter primum & ſecundum, cujus diſtantia a primo eſt tertia pars intervalli communis; pone $b = \frac{4}{3}a$, $c = \frac{7}{3}b$, $d = \frac{10}{3}c$, $e = \frac{13}{3}d$, &c. atque a, b, c, d, e, &c. erunt inter medii quorum quiſque diſtat tertia parte intèrvalli communis a primario ipſum antecedente, vel duabus tertiis partibus ejuſdem intervalli a primario ipſum immediate conſequente. Conſtat igitur relationem intermediorum dari ex Interpolatione relationis primariorum: & ea ſemper eſt interpolabilis, quoniam aſſignatur per Æquationem ad Seriem.

Exemplum III.

Proponatur Series $1, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{35}{128}$, &c. quæ producitur per multiplicationem continuam numerorum $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}$, &c. & ſit a Terminus in medio inter primum & ſecundum, atque ponatur $b = \frac{2}{3}a$, $c =$

$c = \frac{4}{5}b$, $d = \frac{6}{7}c$, $e = \frac{8}{9}d$, &c. tum erunt b, c, d, e, &c. reliqui intermedii qui conſiſtunt in medio inter duos quoſque primarios.

Scholion.

Si Æquatio ad Seriem involvat tres Terminos, debent dari duo; & tres ſi involvat quatuor, & ſic porro, ut habeantur reliqui intermedii. Hujus autem generis eſt Propoſitio *Newtoni* ſeptima in Tractatu de quadratura Curvarum, quæ tamen non tantum obtinet in Curvis ſed & in Serie quacunque. Atque uſui venit hoc Theorema quotieſcunque quæritur Terminus intermedius qui conſiſtit prope initium Seriei: eo enim in caſu ejus valor proveniet in Serie lentiſſime convergente; quare primo quærendus eſt reſpectivus intermedius ſatis remotus a principio, cujus utique valor prodibit in Serie cito approximante: dein ex hoc dato regrediendum eſt ad illum primo propoſitum per relationes Terminorum, ut in hâc Propoſitione.

Propositio XVII.

Omnis Series eſt interpolabilis, cujus Termini conſtant ex factoribus interpolabilibus.

Sit $A \times a \times \alpha$, $B \times b \times \beta$, $C \times c \times \gamma$, $D \times d \times \delta$, &c. Series cujus Termini conſlantur ex tribus factoribus: dico eam eſſe interpolabilem, ſi interpolabiles ſint tres factorum Series, ſcilicet A, B, C, D, &c. a, b, c, d, &c. & α, β, γ, δ, &c.

Quoniam enim Termini intermedii in Serie compoſita, eodem modo formantur ex intermediis correſpondentibus in Seriebus ſimplicibus, ac primarii ex primariis; iidem invenientur ducendo reſpectivos intermedios Serierum ſimplicium in ſe mutuo. Ut ſi T ſit Terminus inter B & C, t reſpectivus inter b & c, atque τ reſpectivus inter β & γ; tum Terminus correſpondens in Serie compoſita, ſcilicet is inter $B \times b \times \beta$ & $C \times c \times \gamma$, erit factum ſub illis tribus, nempe $T \times t \times \tau$. Et ſi plures vel pauciores ſint factores, ſimiliter demonſtrabitur Propoſitio. Q. E. D.

Corollarium. Hinc in Serie interpolanda, ſi duæ vel plures Series factorum ſint accurate interpolabiles, poſſunt rejici ex computo: dein reliquæ ſunt interpolandæ per methodos poſtea tradendas. Nam interpolatio non eſt temere ſuſcipienda; ſed ante exordium operis inquirendum eſt quænam ſit Series ſimpliciſſima ex cujus intercalatione pendet ea Seriei propoſitæ.

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ſitæ. Atque hæc præparatio eſt magna ex parte omnino neceſſaria, ut deveniamus ad concluſiones concinnas & elegantes.

Exemplum I.

Si Series intercalanda ſit $1, \frac{1}{6}x, \frac{3}{40}x^2, \frac{5}{112}x^3, \frac{35}{1152}x^4$, &c. primum reſolvo eandem in tres Series factorum ſimplicium ad modum ſequentem

$$x^0, \quad x^1, \quad x^2, \quad x^3, \quad x^4, \quad \&c.$$

$$1, \quad \frac{1}{3}, \quad \frac{1}{5}, \quad \frac{1}{7}, \quad \frac{1}{9}, \quad \&c.$$

$$1, \quad \frac{1}{2}, \quad \frac{3}{8}, \quad \frac{5}{16}, \quad \frac{35}{128}, \quad \&c.$$

& harum trium Serierum palam eſt duas primas eſſe interpolabiles, non item tertiam; quæ tamen quum ſit ſimplicior, facilius interpolabitur quam ea quæ proponitur. Ut ſi deſideretur is in medio inter ſecundum & tertium Seriei compoſitæ; patet correſpondentes Terminos in prima & ſecunda Serierum ſimplicium eſſe $x^{\frac{3}{2}}$ & $\frac{1}{4}$ reſpective: in tertia Terminus correſpondens appelletur T, dein factum ſub hiſce tribus $x^{\frac{3}{2}}$, $\frac{1}{4}$, & T, ſive $\frac{1}{4}\,T\,x\sqrt{x}$ erit is qui deſideratur. Adeoque Interpolatio Seriei compoſitæ reducitur ad Interpolationem Seriei ſimplicioris.

Exemplum II.

Si detur Series hujuſmodi $1, \frac{r}{p}A, \frac{r+1}{p+1}B, \frac{r+2}{p+2}C, \frac{r+3}{p+3}D$, &c. interpolare licet Series Numeratorum & Denominatorum ſeorſim, hoc eſt, Series

$$1, \quad r, \quad r.\overline{r+1}, \quad r.\overline{r+1}.\overline{r+2}, \quad \&c.$$

$$1, \quad p, \quad p.\overline{p+1}, \quad p.\overline{p+1}.\overline{p+2}, \quad \&c.$$

Deinde Terminus quilibet in Serie Numeratorum diviſus per reſpectivum in eâ Denominatorum, dabit correſpondentem in Serie propoſitâ. Si differentia inter r & p ſit numerus parvus, non opus eſt hujus artificii. Cæterum quando prædicta differentia eſt magna, neceſſe eſt interpolare Numeratores & Denominatores ſeorſim.

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SCHOLION.

Huc etiam referri debent plurimæ hujuſmodi præparationes. Exempli gratiâ, ſi Seriei

$$\&c.\ e,\ d,\ c,\ b,\ a,\quad A,\ B,\ C,\ D,\ E,\ \&c.$$

hinc inde excurrentis in infinitum deſideretur Terminus in medio inter duos medios primarios a & A. Duc Terminos primarios æqualiter a medio diſtantes in ſe mutuo; hoc eſt, A in a, B in b, &c. & componetur nova Series utrinque excurrens in infinitum

$$\&c.\ Dd,\ Cc,\ Bb,\ Aa,\quad Aa,\ Bb,\ Cc,\ Dd,\ \&c.$$

cujus Termini a medio æqualiter hinc inde remoti ſunt inter ſe æquales: & cujus Terminus in medio inter Aa & Aa erit quadratum Termini quæſiti qui conſiſtit in medio inter a & A in Serie prius propoſita. Igitur Terminus ille quæſitus erui poſſit per interpolationem alterutrius Seriei.

Notandum eſt Terminum deſideratum conſiſtere poſſe in diverſis Seriebus, & ex eâ conſideratione facilius nonnunquam inveniri. Ut ſi Terminus quilibet quæſitus lateat in medio inter primum & ſecundum in utrâvis Serierum ſequentium

$$1,\ r,\ r.\overline{r+1},\ r.\overline{r+1}.\overline{r+2},\ \&c.$$

$$1,\ \frac{1}{p},\ \frac{1}{p.\overline{p+1}},\ \frac{1}{p.\overline{p+1}.\overline{p+2}},\ \&c.$$

Tum ducendo Terminos reſpectivos in ſe mutuo producetur nova Series

$$1,\ \frac{r}{p}A,\ \frac{r+1}{p+1}B,\ \frac{r+2}{p+2}C,\ \frac{r+3}{p+3}D,\ \&c.$$

inter cujus Terminum primum & ſecundum is qui medium locum tenet, æqualis eſt quadrato ejus primo propoſiti.

Nonnunquam etiam interpolatio proſpere ſuccedet per Logarithmos, præſertim ſi Terminorum permagnæ ſint differentiæ. Sed hæc & ſimilia addiſcenda ſunt experientiâ. Nam ſicuti in Algebrâ vulgari tota ars Analyſtæ non conſiſtit in reſolutione Æquationum affectarum, ſed in perducendo Problemata ad eaſdem; ita etiam in hac Analyſi, minus ſolertiæ requiritur ad reſolutionem Æquationum differentialium aut interpolationem Serierum; in inveniendis enim Seriebus quæ determinant quantitates incognitas, quæque idoneæ ſint interpolationi, conſiſtit longe major difficultas.

PROPO-

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Propositio XVIII.

Si Termini duarum Serierum formentur ducendo in ſe continue Fractiones quarum Numeratores & Denominatores increſcant perpetuâ additione unitatis, & ſi iidem ſint Numeratores in utrâque; Dico Terminum unius cujus diſtantia ab initio eſt differentia factorum in altera æqualem eſſe Termino hujus cujus diſtantia ab initio eſt differentia factorum in illa Serie, modo primi Termini ſibi invicem æquentur.

Sint duæ Series quarum primi Termini A & a ſibi mutuo æquentur

$$B=\frac{r}{p}A,\ C=\frac{r+1}{p+1}B,\ D=\frac{r+2}{p+2}C,\ E=\frac{r+3}{p+3}D,\ \&c.$$

$$b=\frac{r}{q}a,\ c=\frac{r+1}{q+1}b,\ d=\frac{r+2}{q+2}c,\ e=\frac{r+3}{q+3}d,\ \&c.$$

& in quibus iidem ſunt Numeratores, dico Terminum prioris cujus diſtantia ab initio æqualis eſt differentiæ factorum in poſteriore, nempe $q-r$, æqualem eſſe Termino poſterioris, cujus diſtantia ab initio eſt $p-r$, ſcilicet differentiæ factorum in Serie priore. Et notandum, ubi $p-r$ vel $q-r$ eſt quantitas negativa, Terminos de quibus agitur conſiſtere ante primos hiſce intervallis.

Aſſumamus Terminum quemlibet Seriei prioris, verbi gratiâ ſeptimum, poſitâ $A=1$, ſcilicet

$$G=1\times\frac{r}{p}\times\frac{r+1}{p+1}\times\frac{r+2}{p+2}\times\frac{r+3}{p+3}\times\frac{r+4}{p+4}\times\frac{r+5}{p+5},\ \&c.$$

Et primo ſi ſit $p-r=0$, ſive $p=r$; erit $p+1=r+1$, $p+2=r+2$, &c. adeoque omnes Numeratores & Denominatores ſeſe mutuo deſtruunt, & reſtat $G=1$.

Si ſit $p-r=1$, erit $p=r+1$; & inde $p+1=r+2$, $p+2=r+3$, &c. quo in caſu evaneſcent omnes Numeratores præter primum, & Denominatores omnes præter ultimum; exiſtente $G=1\times\frac{r}{p+5}$, vel $G=1\times\frac{r}{r+6}$ propter $p+5$ æqualem $r+6$.

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Si ſit $p-r=2$, vel $p=r+2$; erit $p+1=r+3$, $p+2=r+4$, &c. & nunc evaneſcent Numeratores omnes præter duos primos, & omnes Denominatores præter duos ultimos, manente $G=1\times\frac{r}{p+4}\times\frac{r+1}{p+5}$, vel $G=1\times\frac{r}{r+6}\times\frac{r+1}{r+7}$, propter $p+4=r+6$, & $p+5=r+7$.

Et ſimiliter ſi ſit $p-r=3$, vel $p=r+3$; evaneſcent Numeratores omnes præter tres primos, & Denominatores omnes præter tres ultimos: & in illo caſu erit $G=1\times\frac{r}{r+6}\times\frac{r+1}{r+7}\times\frac{r+2}{r+8}$. Atque univerſaliter in valore Termini G, tot ſemper erunt Numeratores totidemque Denominatores quot ſunt unitates in $p-r$, ut in Tabulâ ſequente

$p-r=0$, $G=1$,

$p-r=1$, $G=1\times\frac{r}{r+6}$,

$p-r=2$, $G=1\times\frac{r}{r+6}\times\frac{r+1}{r+7}$,

$p-r=3$, $G=1\times\frac{r}{r+6}\times\frac{r+1}{r+7}\times\frac{r+2}{r+8}$,

$p-r=4$, $G=1\times\frac{r}{r+6}\times\frac{r+1}{r+7}\times\frac{r+2}{r+8}\times\frac{r+3}{r+9}$,

&c.

Adeoque ſi ponatur $q=r+6$, ſive $q-r=6$; Terminus hujus Seriei 1, $\frac{r}{p}$ A, $\frac{r+1}{p+1}$ B, &c. cujus intervallum ab initio eſt $q-r$ ſive 6, æqualis eſt Termino hujus 1, $\frac{r}{q}a$, $\frac{r+1}{q+1}b$, &c. cujus intervallum ab initio eſt $p-r$. Et eodem modo manifeſta erit Propoſitio in aliis caſibus. Q. E. D.

Corollarium. Hinc ſi differentia factorum p & r non ſit permagna, Terminus Seriei prioris utcunque longe remotus ab initio ſemper determinabitur per Terminum in Serie poſteriore qui parum diſtat a principio. Ut Exemplis ſequentibus patebit.

Exemplum

Exemplum I.

Sit $r=3$, $p=5$, $q=10$; & hiſce valoribus ſubſtitutis, evadent duæ Series

$$1,\ \frac{3}{5},\ \frac{3.4}{5.6},\ \frac{3.4.5}{5.6.7},\ \frac{3.4.5.6}{5.6.7.8},\ \&c.$$

$$1,\ \frac{3}{10},\ \frac{3.4}{10.11},\ \frac{3.4.5}{10.11.12},\ \frac{3.4.5.6}{10.11.12.13},\ \&c.$$

Eſt vero $q-r=7$, $p-r=2$; adeoque Terminus in priore cujus intervallum ab initio eſt 7, æqualis erit Termino in poſteriore, cujus intervallum ab initio eſt 2; ſive quod idem eſt, octavus Terminus Seriei prioris $\frac{3.4.5.6.7.8.9}{5.6.7.8.9.10.11}$ æqualis eſt tertio poſterioris $\frac{3.4}{10.11}$. Et notandum eſt ubi differentia inter p & r eſt numerus integer, tum Terminum quemvis Seriei prioris ſemper æquari primario alicui in poſteriore.

Exemplum II.

Sit prior Series 1, $\frac{2}{1}$ A, $\frac{4}{3}$ B, $\frac{6}{5}$ C, $\frac{8}{7}$ D, &c. & quoniam factorum incrementum eſt binarius, divide Numeratores & Denominatores per binarium; & prodibit Series 1, $\frac{1}{\frac{1}{2}}$ A, $\frac{2}{\frac{1}{2}+1}$ B, $\frac{3}{\frac{1}{2}+2}$ C, &c. ubi jam incrementum factorum eſt unitas; & per conſequens hæc Series comparari poſſit cum illâ in Theoremate, prodeuntibus $r=1$, $p=\frac{1}{2}$: Sit etiam m intervallum inter primum Terminum Seriei & quemvis alium; & erit $m=q-r$, vel $m=q-1$, atque $q=m+1$; quo ſubſtituto evadet Series poſterior

$$1,\ \frac{a}{m+1},\ \frac{2b}{m+2},\ \frac{3c}{m+3},\ \frac{4d}{m+4},\ \&c.$$

cujus Terminus ab initio remotus intervallo $p-r$, ſive $-\frac{1}{2}$, æqualis erit Termino prioris cujus intervallum ab initio eſt quantitas quæcunque m. Hoc eſt, Terminus prioris utcunque longe remotus a principio, ſcilicet diſtantiâ utcunque magnâ m ſemper æquabitur Termino poſterioris, qui conſiſtit ante primum dimidio intervalli communis.

 Vel

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Vel ſi ſumatur Series 1, $\frac{1}{2}$ A, $\frac{3}{4}$ B, $\frac{5}{6}$ C, $\frac{7}{8}$ D, &c. cujus Termini ſunt reciproci ejus de quâ nunc egimus, erit $r=\frac{1}{2}$, $p=1$; & ſi m ſit intervallum inter Terminum primum & quemvis alium, erit $m=q-r$, vel $m=q-\frac{1}{2}$, ac $q=m+\frac{1}{2}$, & poſterior Series evadet

$$1, \frac{a}{2m+1}, \frac{3b}{2m+3}, \frac{5c}{2m+5}, \frac{7d}{2m+7}, \&c.$$

In qua Terminus ab initio remotus diſtantiâ $p-r$, ſive $\frac{1}{2}$; id eſt, Terminus in medio inter primum & ſecundum, æqualis erit Termino cuicunque prioris cujus diſtantia ab initio eſt quantitas quælibet m. Ut ſi deſideretur Terminus Seriei milleſimus primus, cujus utique intervallum ab initio eſt mille, erit $m=1000$, & Terminus Seriei

$$1, \frac{a}{2001}, \frac{3b}{2003}, \frac{5c}{2005}, \frac{7d}{2007}, \&c.$$

qui conſiſtit iu medio inter primum 1 & ſecundum $\frac{1}{1999}$ æqualis erit milleſimo primo Termino hujus 1, $\frac{1}{2}$ A, $\frac{3}{4}$ B, $\frac{5}{6}$ C, $\frac{7}{8}$ D, &c. Cujus etiam Termini intermedii eodem modo inveniuntur; nam ſi pro m ſcribatur $999\frac{1}{2}$, proveniet Series

$$1, \frac{a}{2000}, \frac{3b}{2002}, \frac{5c}{2004}, \frac{7d}{2006}, \&c.$$

cujus Terminus in medio inter primum & ſecundum, æqualis eſt Termino hujus Seriei 1, $\frac{1}{2}$ A, $\frac{3}{4}$ B, $\frac{5}{6}$ C, $\frac{7}{8}$ D, &c. qui conſiſtit in medio inter milleſimum & milleſſimum primum.

Exemplum III.

Si quæratur Terminus hujus Seriei 1, $\frac{2}{1}$ A, $\frac{5}{4}$ B, $\frac{8}{7}$ C, $\frac{11}{10}$ D, &c. cujus intervallum a principio ſit quantitas quælibet m: primo divide Numeratores & Denominatores per incrementum ipſorum commune 3, & evadet Series 1, $\frac{\frac{2}{3}}{\frac{1}{3}}$ A, $\frac{\frac{5}{3}}{\frac{4}{3}}$ B, &c. adeoque erit $r=\frac{2}{3}$, $p=\frac{1}{3}$, $q-$

$r=$

$r = q - \frac{2}{3} = m$; & inde $q = m + \frac{2}{3}$: unde Series poſterior evadet

$$1,\ \frac{2a}{3m+2},\ \frac{5b}{3m+5},\ \frac{8c}{3m+8},\ \frac{11d}{3m+11},\ \&c.$$

cujus Terminus ab initio diſtans intervallo $p - r$, ſive $-\frac{1}{3}$, id eſt, Terminus qui conſiſtit ante primum tertiâ parte intervalli communis, æqualis eſt Termino Seriei prioris, qui diſtat a principio, intervallo quantumvis magno m.

De Quantitatum Differentiis.

Sit a, b, c, d, e, Series quotcunque quantitatum, & ſi priores auferantur de poſterioribus, manebunt Differentiæ primæ $b-a$, $c-b$, $d-c$, $e-d$: dein ſi harum Differentiarum priores ſimiliter auferantur de poſterioribus, relinquentur Differentiæ ſecundæ $c-2b+a$, $d-2c+b$, $e-2d+c$; quarum rurſus Differentiæ conſtituunt Differentias tertias $d-3c+3b-a$, $e-3d+3c-b$, quantitatum a, b, c, d, e. Et ſic porro proceditur uſque dum perventum fuerit ad ultimam Differentiam ut in Tabulâ ſequente.

	1mæ	2dæ	3tiæ	4ta
a				
	$b-a$			
b		$c-2b+a$		
	$c-b$		$d-3c+3b-a$	
c		$d-2c+b$		$e-4d+6c-4b+a$
	$d-c$		$e-3d+3c-b$	
d		$e-2d+c$		
	$e-d$			
e				

Sit $1-x$ binomium in quo unciæ $+1$, -1 eædem ſint ac Coefficientes in Differentiis primis; tum unciæ quadrati $1-2x+x^2$, ſcilicet $+1$, -2, $+1$ erunt Coefficientes in Differentiis ſecundis; item unciæ Cubi 1, -3, $+3$, -1 erunt Coefficientes in Differentiis tertiis: & in genere Coefficientes in quolibet ordine Differentiarum erunt unciæ in correſpondente dignitate binomii. Atque hiſce præcognitis, pergere licet per ſaltum ad quemlibet ordinem Differentiarum abſque conſideratione intermediarum.

Duæ quantitates habent Differentiam primam, tres habent ſecundam, quatuor tertiam; nec ulteriores habere poſſunt. Sed nonnunquam accidit

cidit quendam ordinem Differentiarum conſtituere progreſſionem æqualium, quo in caſu ulteriores Differentiæ non habentur, utcunque magnus ſit quantitatum numerus. Sic in progreſſione arithmeticâ æquantur differentiæ primæ, adeoque non dantur ſecundæ. Et in Serie quadratorum 1, 4, 9, 16, 25, &c. quorum radices ſunt æquidifferentes, differentiæ primæ 3, 5, 7, 9, &c. ſunt in progreſſione arithmeticâ, ſecundæ ſunt æquales, câque de cauſâ tertiæ ſunt nihil. Sic etiam in Serie cuborum 1, 8, 27, 64, 125, 216, &c. Differentiæ primæ ſunt 7, 19, 37, 61, 91, &c. ſecundæ 12, 18, 24, 30, &c. tertiæ 6, 6, 6, &c. utique æquales; adeoque quartæ nihil.

Et univerſaliter ſint A, B, C, D, E, quantitates quotcunque datæ, z vero variabilis, tum in expreſſione $A + Bz + Cz^2 + Dz^3 + Ez^4$ ſcribe numeros quoſvis æquidifferentes ſucceſſive pro z; & quantitatum provenientium ultimæ Differentiæ determinabuntur per altiſſimam dignitatem z^4, nullâ ratione habitâ ad inferiores. Sic in hoc caſu quarta differentia eſt ultima, propterea quod quarta poteſtas z^4 ſit hic altiſſima.

Sæpiſſime differentiæ conſtituunt Seriem convergentem in caſibus quando non abrumpunt. Ut ſi $a, b, c, d, e,$ &c. ſint fere inter ſe æquales, earumque Differentiæ primæ $b-a,\ c-b,\ d-c,\ e-d,$ &c. etiam ſibi mutuo fere æquales, atque etiam ſecundæ ſimiliter & ſequentes ſibi mutuo quamproxime æquales, tum $a,\ b-c,\ c-2b+a,\ d-3c+3b-a,$ &c. conſtituent Seriem convergentem. Item differentiæ Terminorum quorum relatio definitur Æquatione

$$T \times \overline{z^{\theta} + az^{\theta-1} + bz^{\theta-2} + \&c.} = T' \times \overline{z^{\vartheta} + cz^{\vartheta-1} + dz^{\vartheta-2} + \&c.}$$

ſumptæ ad modum præcedentem conficient Seriem convergentem. Sed non expectandum eſt Differentias quantitatum quarumcumque vel convergere vel abrumpere. Hoc tantum accidit in iis quantitatibus quæ increſcunt aut decreſcunt accurate vel quamproxime eâdem celeritate ac certæ quædam dignitates numerorum æquidifferentium.

De Deſcriptione Curvarum per data puncta.

Newtonus in Epiſtolâ ad *Oldenburgum* Anno 1676 datâ, poſtquam monſtraverat artificium evitandi Series nimium complexas in quadraturâ Curvarum trinomialium; inquit, " Sed hæc minoris facio, quod ubi " Series ſimplices non ſunt ſatis tractabiles, aliam nondum communi- " catam methodum habeo, quâ pro libitu acceditur ad quæſitum. " Ejus fundamentum eſt commoda, expedita, generalis ſolutio hujus " Problematis, *Curvam Geometricam deſcribere, quæ per data quotcunque* " *puncta tranſibit.* Docuit *Euclides* deſcriptionem Circuli per tria pun- " cta. Poteſt etiam Sectio Conica deſcribi per quinque data puncta:

" &

" & Curva trium dimenſionum per ſeptem data puncta; (adeo ut in
" poteſtate habeam deſcriptionem omnium Curvarum iſtius ordinis, quæ
" per ſeptem tantum puncta determinantur.) Hæc ſtatim Geometrice
" fiunt nullo calculo interpoſito. Sed ſuperius Problema eſt alterius
" generis: & quamvis primâ fronte intractabile videatur; tamen res
" aliter ſe habet. Eſt enim fere ex pulcherrimis quæ ſolvere deſi-
" derem.

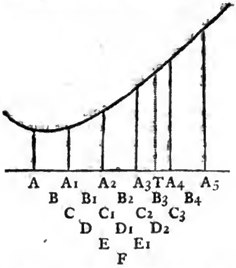
Newtonus in Propoſitione ſexageſimâ *Arithmeticæ Univerſalis*, docet deſcriptionem Parabolæ Conicæ per quatuor puncta: vel potius docet methodum inveniendi Æquationem ad Parabolam quæ tranſibit per quatuor data puncta. Et eâdem methodo deſcribere licet Lineam tertii ordinis per novem puncta, & Lineam quarti per quatuordecem. Et ſic in reliquis. Sed noſtrum inſtitutum non requirit ſolutionem adeo generalem; ſufficit enim deſcribere figuram Parabolicam per extremitates quotcunque ordinatarum quæ ſunt ſibi mutuo, & etiam Axi Curvæ parallelæ. Sed neque deſcriptio Curvæ organica motu angulorum, aut alio quovis modo, ad præſens propoſitum conducit: eodem res redit ſive Curva actu deſcribatur, ſive deſcripta concipiatur. Curvæ enim hic nullatenus ſunt neceſſariæ, niſi quatenus adjumento ſunt intellectui in rite concipiendo Problemate. Etenim deſcriptio Parabolæ per data puncta, idem eſt omnino Problema ac aſſignatio quantitatum ex datis ipſarum Differentiis; quæ per Algebram ſolam, idque per reſolutionem Æquationum ſimplicium ſemper perficitur.

PROPOSITIO XIX.

Detur Series Ordinatarum æquidiſtantium ex una tantum parte pergens in infinitum, & oporteat invenire Lineam Parabolicam quæ tranſibit per extremitates omnium.

Deſignent A, A1, A2, A3, A4, &c. Ordinatas æquidiſtantes inſiſtentes Abſciſſæ normaliter. Collige earum differentias primas B, B1, B2, B3, &c. ſecundas C, C1, C2, &c. tertias D, D1, &c. & ſic porro. Adeo ut A ſit Ordinata prima, B differentia prima duarum primarum Ordinatarum, C differentia ſecunda trium primarum Ordinatarum, D differentia tertia quatuor primarum & ſic deinceps. Differentiæ autem colligi debent auferendo priores ubique de poſterioribus; hoc eſt, ponendo $B = A1 - A$, $B1 = A2 - A1$, &c. $C = B1 - B$, &c. & ita porro in reliquis; ponendo eas eſſe negativas quæ oriuntur ex ſubductione quantitatis majoris de minore.

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Hiſce præmiſſis, ſit T quævis Ordinata in genere, primaria vel intermedia, cujus diſtantia a primâ Ordinatâ A, nempe AT ſit ad intervallum commune æquidiſtantium ut quantitas indeterminata z ad unitatem, eritque

$$\begin{aligned} T = A &+ \\ &B \times \frac{z}{1} + \\ &C \times \frac{z}{1} \times \frac{z-1}{2} + \\ &D \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} + \\ &E \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} + \\ &F \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} \times \frac{z-4}{5} + \\ &\&c. \end{aligned}$$

Hic eſt valor Ordinatæ cujuſvis T jacentis ad eaſdem partes Ordinatæ primæ A, ad quas jacent reliquæ: quod ſi jaceret ad alteras, tum mutandum foret ſignum Abſciſſæ z. Etenim Abſciſſam pono affirmativam quæ ab initio tendit dextram verſus inter Ordinatas poſteriores; negativam vero quæ ad contrarias partes porrigitur. Propoſitio autem ſic demonſtratur.

1 Concipe

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Concipe Ordinatam T motu parallelo latam deferri ſuper Abſciſſam, adeo ut ſucceſſive deveniat ad loca reliquarum. Et quoniam illius diſtantia ab Ordinatâ primâ ponitur eſſe ad intervallum Ordinatarum commune ut z ad unitatem; erit z ſucceſſive æqualis 0, 1, 2, 3, 4, &c. & interea T æqualis Ordinatis A, A1, A2, A3, &c. per vices, cuique in ſuo loco. Igitur ad eruendas Coefficientes A, B, C, D, &c. quæ efficiunt Parabolam tranſire per extremitates Ordinatarum, in Æquatione ad figuram $T = A + B\frac{z}{1} + C\frac{z}{1}\times\frac{z-1}{2} + D\frac{z}{1}\times\frac{z-1}{2}\times\frac{z-2}{3} +$ &c. ſcribe Ordinatas A, A1, A2, A3, &c. ſucceſſive pro T; & interea pro z, longitudines Abſciſſæ ordine ſequentes, id eſt, 0, 1, 2, 3, &c. atque emergent Æquationes

$A = A,$	Unde $A = A,$
$A1 = A + B,$	$B = A1 - A,$
$A2 = A + 2B + C,$	$C = A2 - 2A1 + A,$
$A3 = A + 3B + 3C + D,$	$D = A3 - 3A2 + 3A1 - A,$
$A4 = A + 4B + 6C + 4D + E,$	$E = A4 - 4A3 + 6A2 - 4A1 + A,$
&c.	&c.

Quippe ex valoribus Ordinatarum A, A1, A2, &c. viciſſim eruuntur valores Coefficientium A, B, C, &c. Ex quibus patet Ordinatam primam A eſſe primam Coefficientem; item differentiam duarum primarum Ordinatarum eſſe Coefficientem ſecundam; & differentiam ſecundam primarum trium eſſe Coefficientem tertiam; & ſic in infinitum. Igitur valores Coefficientium in ſolutione poſiti efficiunt Parabolam tranſire per extremitates Ordinatarum. Q. E. D.

Idem aliter.

Fingamus eſſe in genere $T = A + B\frac{z}{1} + C\frac{z}{1}\times\frac{z-1}{2} + D\frac{z}{1}\times\frac{z-1}{2}\times\frac{z-2}{3} +$ &c. ubi A, B, C, D, &c. ſunt Coefficientes determinandæ. Scribe valores variabilium conſequentes T', $z+1$ pro antecedentibus T, z; & emerget $T' = A + B\frac{z+1}{1} + C\frac{z+1}{1}\times\frac{z}{2} + D\frac{z+1}{1}\times\frac{z}{2}\times\frac{z-1}{3} +$ &c. de quo ſi ſubducatur valor ipſius T, habebitur

T' =

$$T' - T = B + C\frac{z}{1} + D \times \frac{z}{1} \times \frac{z-1}{2} + \&c.$$

Ubi ſubſtituendo T'', T', $z+1$ pro T', T, z, prodibit $T'' - T' = B + C\frac{z+1}{1} + D\frac{z+1}{1} \times \frac{z}{2} + \&c.$ a quo valor ipſius $T' - T$ ſubductus relinquit

$$T'' - 2T' + T = C + D\frac{z}{1} + \&c.$$

Et ſimiliter invenies $T''' - 3T'' + 3T' - T = D + \&c.$ Deſignet jam T primam Ordinatam, & valor correſpondens Abſciſſæ erit nihil; quo ſubſtituto invenies

$$A = T,$$
$$B = T' - T,$$
$$C = T'' - 2T' + T,$$
$$D = T''' - 3T'' + 3T' - T,$$
$$\&c.$$

Hoc eſt, prima Coefficiens A æqualis eſt Ordinatæ primæ T; ſecunda B æqualis eſt differentiæ inter duas primas Ordinatas T & T′; tertia C æqualis eſt differentiæ ſecundæ trium primarum Ordinatarum, T, T′, T″; quarta C æqualis eſt differentiæ tertiæ primarum quatuor Ordinatarum; & ſic porro in reliquis ut jam demonſtratum eſt.

Exemplum I.

Dentur quinque Ordinatæ 1, 4, 2, 3, 9, per quarum extremitates oportet Parabolam tranſire. Collige earum differentias primas 3, —2, 1, 6; ſecundas —5, 3, 5, tertias 8, 2; & ultimam —6. Dein juxta præſcripta in ſolutione Propoſitionis, pro A, B, C, &c. ponantur Ordinata prima, & prima quæque differentia reſpective; hoc eſt, A = 1, B = 3, C = —5, D = 8, E = —6; at F, G, &c. erunt nihil. Hiſce autem valoribus ſubſtitutis, prodit Æquatio ad Parabolam

1		4		2		3		9
	3		—2		1		6	
		—5		3		5		
			8		2			
				—6				

$$T = 1 + 3\frac{z}{1} - 5\frac{z}{1} \times \frac{z-1}{2} + 8\frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} - 6\frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4},$$

quæ in ordinem reducta fit $T = \frac{12 + 116z - 111z^2 + 24z^3 - 3z^4}{12}$. Atque

que ad probandum opus, ſcribe 0, 1, 2, 3, 4, ſucceſſive pro Abſciſſa z, & pro T provenient Ordinatæ quinque propoſitæ.

Ceterum Ordinatæ poſſunt ſumi in ordine inverſo, modo mutentur ſigna differentiarum in ordinibus alternis: tunc ponendum eſt $A = 9$, $B = -6$, $C = 5$, $D = -2$, $E = -6$; quibus ſcriptis & Æquatione prodeunte in ordinem redacta, tandem obtinebitur $T =$ $\frac{108 - 92z + 9z^2 + 14z^3 - 3z^4}{12}$; in quâ ſi pro z ſubſtituantur 0, 1, 2, 3, 4, prodibunt Ordinatæ propoſitæ in ordine inverſo. Atque hic obtinentur duæ Æquationes pro eâdem Parabolâ, quoniam Abſciſſa initium ducit nunc a primâ, nunc ab ultimâ Ordinatâ.

EXEMPLUM II.

Oporteat nunc invenire Æquationem ad Parabolam quæ tranſit per extremitates ſex Ordinatarum æquidiſtantium 5, 3, 7, 23, 57, 115. Collige earum differentias primas, reliquaſque uſque dum perventum fuerit ad ultimas, ut in margine. Et invenies $A = 5$, $B = -2$, $C = 6$, $D = 6$. Quibus ſubſtitutis, oritur

5	3	7	23	57	115
−2	4	16	34	58	
6	12	18	24		
6	6	6			

$$T = 5 - 2\frac{z}{1} + 6\frac{z}{1} \times \frac{z-1}{2} + 6\frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3},$$

quæ reducta fit $T = 5 - 3z + z^3$. Et ſi pro z ſcribas ſucceſſive 0, 1, 2, 3, 4, 5, provenient ſex Ordinatæ propoſitæ.

Linea recta tranſit per duo puncta, Parabola Conica per tria, Cubica per quatuor, Biquadratica per quinque, & ſic in infinitum. Nonnunquam vero accidit Curvam inferioris ordinis tranſire per plura puncta, ut in Exemplo noviſſimo. Ordo autem Parabolæ ſemper denotatur per ultimum Ordinem differentiarum. Quod ſi numerus Ordinatarum ſit infinitus, & progreſſio æqualium differentiarum non obveniat, in eo inquam caſu Curva erit dimenſionum infinitarum, valore ipſius T excurrente in Seriem infinitam.

SCHOLION.

In hac ſolutione poſuimus diſtantiam communem Ordinatarum eſſe unitatem; verum ſi pro eadem uſurpaſſem quantitatem quamvis n, prodiiſſet

$$T = A + B \times \frac{z}{n} + C \times \frac{z}{n} \times \frac{z-n}{2n} + D \times \frac{z}{n} \times \frac{z-n}{2n} \times \frac{z-2n}{3n} + \&c.$$

Finge jam Ordinatam ſecundam $A1 = A + \dot{A}n$,

tertiam $A2 = A + 2\dot{A}n + \ddot{A}n^2$,

quartam $A3 = A + 3\dot{A}n + 3\ddot{A}n^2 + \dddot{A}n^3$,

quintam $A4 = A + 4\dot{A}n + 6\ddot{A}n^2 + 4\dddot{A}n^3 + \ddddot{A}n^4$,

&c.

Et invenies viciſſim $B = \dot{A}n$, $C = \ddot{A}n^2$, $D = \dddot{A}n^3$, $E = \ddddot{A}n^4$, &c. quibus valoribus ſubſtitutis pro B, C, D, E, &c. orietur

$$T = A + \dot{A}\frac{z}{1} + \ddot{A}\frac{z}{1} \times \frac{z-n}{2} + \dddot{A}\frac{z}{1} \times \frac{z-n}{2} \times \frac{z-2n}{3} + \&c.$$

Evadat jam intervallum commune n nihil, & $\dot{A}$, $\ddot{A}$, $\dddot{A}$, &c. evadent fluxiones Ordinatæ primæ A, dummodo fluxio Abſciſſæ z ſit unitas; atque proveniet

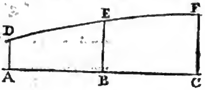
$$T = A + \dot{A}z + \frac{1}{2}\ddot{A}z^2 + \frac{1}{6}\dddot{A}z^3 + \frac{1}{24}\ddddot{A}z^4 + \&c.$$

Igitur coincidentibus Ordinatis æquidiſtantibus, incidimus in Seriem ubi Coefficientes Terminorum ſunt fluxiones Ordinatæ primæ reſpective diviſæ per numeros 1, 2, 6, 24, &c. qui generantur per continuam multiplicationem horum, 1, 2, 3, 4, &c. Et hoc primus deprehendit D. *Taylor* in *Methodo Incrementorum*, & poſtea *Hermanus* in Appendice ad *Phoronomiam*.

Hinc ſi Ordinata Curvæ cujuſcunque reſolvatur in Seriem hujus formæ $A + Bz + Cz^2 + Dz^3 + \&c.$ ubi exponentes Abſciſſæ z ſunt numeri integri & affirmativi; primus Terminus A eſt prima Ordinata, utique quæ tranſit per initium Abſciſſæ; primi duo $A + Bz$ denotant lineam rectam quæ tranſit per duo Curvæ puncta coincidentia, quæque proinde tangit Curvam; primi tres Termini $A + Bz + Cz^2$ definiunt Parabolam Conicam quæ tranſit per tria Curvæ puncta coincidentia, quæque eâ de cauſâ tangit Curvam, eandemque habet Curvaturam in puncto per quod tranſit Ordinata prima; primi quatuor Termini $A + Bz + Cz^2 + Dz^3$ definiunt Parabolam Cubicam quæ tranſit per quatuor Curvæ puncta coeuntia, id eſt, quæ tangit Curvam, eandemque habet Curvaturam & Variationem Curvaturæ in puncto Contactus. Denique tota Series $A + Bz + Cz^2 + Dz^3 + \&c.$ eſt Ordinata Parabolæ dimenſionum infinitarum quæ tangit Curvam, & in puncto Contactus, habet eandem Curvaturam, Variationem Curvaturæ,

Variationem Variationis, & ſic in infinitum, ut exprimitur a *Newtono* in Propoſitione decimâ Libri ſecundi Principiorum. Vel quod eodem redit, tota Series eſt Ordinata Parabolæ tranſeuntis per Ordinatas Curvæ æquidiſtantes, numero infinitas, & coincidentes cum Ordinatâ primâ.

Hinc ideam habemus analogiæ quæ eſt inter Methodum Differentialem & Methodum vulgarem Serierum; hæc procedit per Fluxiones ſive rationes differentiarum ultimas, illa vero generaliter per differentias cujuſcunque magnitudinis.



Deſignet DEF Curuam quamvis, cujus Abſciſſa AC decuſſat Ordinatas æquidiſtantes AD, BE, CF, ad angulos rectos. Sitque $AB = z$, $AD = A$; eritque ex ſeperioribus

$BE = A + \dot{A}z + \frac{1}{2}\ddot{A}z^2 + \frac{1}{6}\dddot{A}z^3 + \frac{1}{24}\ddddot{A}z^4 + \&c.$ ſcilicet hicce valor ipſius BE eſt Ordinata Curvæ Parabolicæ quæ coincidit cum altera Curva in puncto D: igitur pro Area Curvæ uſurpare licet Aream ejuſdem Parabolæ, quæ per methodum Fluxionum inverſam prodit $ABED = Az + \frac{1}{2}\dot{A}z^2 + \frac{1}{6}\ddot{A}z^3 + \frac{1}{24}\dddot{A}z^4 + \frac{1}{120}\ddddot{A}z^5 + \&c.$

Et eâdem prorſus ratione ſi BE dicatur y, exiſtente $AB = BC = z$, erit Area

$$BCFE = yz + \frac{1}{2}\dot{y}z^2 + \frac{1}{6}\ddot{y}z^3 + \frac{1}{24}\dddot{y}z^4 + \frac{1}{120}\ddddot{y}z^5 + \&c.$$

In quâ ſi mutetur ſignum Abſciſſæ z, obtinebitur Area BEDA negative expreſſa, ſcilicet mutando ſignum Abſciſſæ obtinetur Area jacens ad alteras partes Ordinatæ. Sed Area illa affirmative expreſſa evadit

$$BEDA = yz - \frac{1}{2}\dot{y}z^2 + \frac{1}{6}\ddot{y}z^3 - \frac{1}{24}\dddot{y}z^4 + \frac{1}{120}\ddddot{y}z^5 - \&c.$$

Et hæc eſt Series D. *Johannis Bernoullii* exhibens Aream per Ordinatam ultimam ejuſque Fluxiones expreſſam; quamque nos jam dedimus per Fluxiones Ordinatæ primæ. Notandum autem eſt Seriem priorem non extendi ad caſus in quibus Ordinata prima tangit Curvam, & Seriem D. *Bernoullii* non extendi ad eos in quibus Ordinata ultima tangit Curvam. Nam Parabola cujus Area uſurpatur pro Area Curvæ quadrandæ nullam Ordinatarum tangere poteſt; adeoque coincidere nequit cum alterâ Curvâ Ordinatam ſuam tangente. Hujuſmodi enim Expreſſiones pro Areis & Ordinatis Curvarum præſupponunt formam Seriei $A + Bz + Cz^2 + \&c.$ in quâ exponentes Abſciſſæ z ſunt numeri integri & affirmativi.

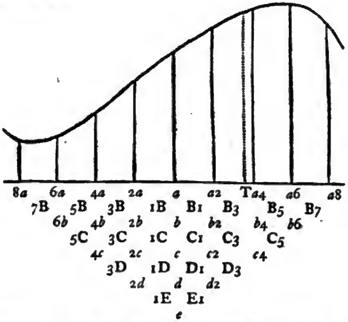
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Propositio XX.

Detur Series Ordinatarum æquidiſtantium utrinque excurrens in infinitum, & oporteat invenire Lineam Parabolicam quæ tranſibit per extremitates omnium.

Caſus primus.

Deſignet a Ordinatam in medio omnium, ſintque $a2$, $a4$, $a6$, $a8$, &c. eæ ex una parte; & $2a$, $4a$, $6a$, $8a$, &c. eæ ex altera; pergente progreſſione utrinque in infinitum. Collige earum differentias primas 7B, 5B, 3B, 1B, B1, B3, B5, B7; ſecundas $6b$, $4b$, $2b$, b, $b2$, $b4$, $b6$; tertias 5C, 3C, 1C, C1, C3, C5; quartas $4c$, $2c$, c, $c2$, $c4$; & ſic porro in reliquis, auferendo ſemper antecedentes de conſequentibus ut in Propoſitione ſuperiore.



Sint jam a, b, c, d, e, &c. Ordinata media & differentiæ in ordinibus alternis reſpective. Sintque 1B & B1, 1C & C1, 1D & D1, 1E & E1, &c. duæ mediæ differentiæ in reliquis ordinibus; & ponantur B =
1B +

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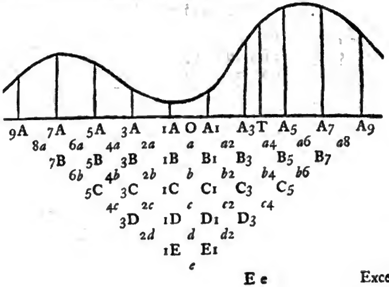
$1B + B1$, $C = 1C + C1$, $D = 1D + D1$, $E = 1E + E1$, &c. Sitque intervallum inter Ordinatam quamvis T & mediam a ad intervallum commune æquidiſtantium ut z ad unitatem; eritque Ordinata

$$T = a + \frac{Bz + bzz}{1.2} + \frac{2Cz + czz}{1.2} \times \frac{zz-1}{3.4} + \frac{3D + dzz}{1.2} \times \frac{zz-1}{3.4} \times \frac{zz-4}{5.6} + \frac{4E + ezz}{1.2} \times \frac{zz-1}{3.4} \times \frac{zz-4}{5.6} \times \frac{zz-9}{7.8} + \frac{5F + fzz}{1.2} \times \frac{zz-1}{3.4} \times \frac{zz-4}{5.6} \times \frac{zz-9}{7.8} \times \frac{zz-16}{9.10} + \&c.$$

Ubi notandum eſt Abſciſſam z eſſe negativam quando Ordinata quæſita T jacet ad contrarias partes Ordinatæ mediæ.

Caſus ſecundus.

Sint jam 1A & A1, Ordinatæ duæ mediæ; A3, A5, A7, A9, &c. eæ ex unâ parte; & 3A, 5A, 7A, 9A, &c. eæ ex alterâ. Collige earum differentias primas $8a$, $6a$, $4a$, $2a$, a, $a2$, $a4$, $a6$, $a8$; ſecundas 7B, 5B, 3B, 1B, B1, B3, B5, B7; tertias $6b$, $4b$, $2b$, b, $b2$, $b4$, $b6$, &c. & ſic deinceps; ſubducendo ubique priores de poſterioribus



E e Excerpe

Excerpe nunc differentias medias $a, b, c, d, e,$ &c. ut & duas medias in aliis ordinibus, ſcilicet $_1A$ & A_1, $_1B$ & B_1, $_1C$ & C_1, $_1D$ & D_1, $_1E$ & E_1, &c. & ponantur $A = {_1A} + A_1$, $B = {_1B} + B_1$, $C = {_1C} + C_1$, $D = {_1D} + D_1$, $E = {_1E} + E_1$, &c. ſit vero O punctum in medio inter Ordinatas duas medias $_1A$ & A_1. Et cujuſvis Ordinatæ T diſtantia a medio puncto, nempe O T ſit ad intervallum commune æquidiſtantium ut z ad Binarium: Eritque

$$T = \frac{A + az}{2} + \frac{3B + bz}{2} \times \frac{zz - 1}{4.6} + \frac{5C + cz}{2} \times \frac{zz-1}{4.6} \times \frac{zz-9}{8.10} + \frac{7D + dz}{2} \times \frac{zz-1}{4.6} \times \frac{zz-9}{8.10} \times \frac{zz-25}{12.14} + \frac{9E + ez}{2} \times \frac{zz-1}{4.6} \times \frac{zz-9}{8.10} \times \frac{zz-25}{12.14} \times \frac{zz-49}{16.18} + \text{\&c.}$$

Et in hoc quoque caſu z eſt affirmativa quando T jacet ad eaſdem partes puncti medii O ac in Schemate, & negativa quando jacet ad contrarias. Uterque autem caſus facillime demonſtratur ad modum Propoſitionis ſuperioris.

Exemplum I.

Dentur quinque Ordinatæ $-3, -8, 1, 12, 37,$ per quarum extremitates ducenda ſit Parabola. Quære earum differentias primas $-5, 9, 11, 25$; ſecundas $14, 2, 14$; tertias $-12, 12$; & ultimam 24. Dein quoniam numerus Ordinatarum eſt impar, procede per Caſum primum. Et inchoando ab Ordinata media, perge ad differentias medias in ordinibus alternis, ponendo $a = 1$, $b = 2$, $c = 24$; dein $B = 9 + 11 = 20$, $C = -12 + 12 = 0$. Et hiſce ſubſtitutis, prodit $T = 1 + \frac{20z + 2zz}{1.2} + \frac{24zz}{1.2} \times \frac{zz-1}{3.4}$, ſive $T = 1 + 10z + z^4$. Eſtque hæc Ordinata Parabolæ quatuor dimenſionum tranſeuntis per extremitates quinque Ordinatarum propoſitarum, ut conſtabit ſcribendo numeros $-2, -1, 0, 1, 2,$ ſucceſſive pro z. In hoc caſu Curva decuſſat

-3		-8		1		12		37
	-5		9		11		25	
		14		2		14		
			-12		12			
				24				

baſin,

baſin, quoniam Ordinatæ ſunt partim negativæ & partim affirmativæ.

Exemplum II.

Dentur jam ſex Ordinatæ 1, 5, 10, 10, 5, 1 per quarum extremitates oportet ducere Parabolam. Quære earum differentias ut in margine: & quoniam numerus Ordinatarum eſt par, adhibeatur Caſus ſecundus. Tum incipiendo ab Ordinatis duabus mediis, & progrediendo ad binas medias differentias, erit $A=10+10=20$, $B=-5-5=-10$, $C=6+6=12$; deinde $a=0$, $b=0$, $c=0$. Quibus valoribus ſubſtitutis, erit $T=\frac{20}{2}-\frac{30}{2}\times\frac{zz-1}{4.6}+\frac{60}{2}\times\frac{zz-1}{4.6}\times\frac{zz-9}{8.10}$; quæ Æquatio in ordinem reducta evadit $T=\frac{689-50z^2+z^4}{64}$. Atque ad probandam operationem, ſcribe $-5, -3, -1, 1, 3, 5$, ſucceſſive pro z, & exibunt Ordinatæ propoſitæ. Etenim in caſu Propoſitionis ſecundo, intervallum commune Ordinatarum, ſive, quod perinde eſt, incrementum Abſciſſæ æquatur Binario.

1	5	10	10	5	1
4	5	0	−5	−4	
1	−5	−5	1		
−6	0	6			
6	6				
0					

In hoc Exemplo deſunt dignitates Abſciſſæ imparium dimenſionum, quoniam Ordinatæ a principio Abſciſſæ hinc inde æqualiter diſtantes ſunt ejuſdem ſigni, & inter ſe æquales. Nam hoc in caſu Æquatio ad Parabolam eadem manet, etiamſi mutetur ſignum Abſciſſæ. Quod ſi Ordinatæ propoſitæ fuiſſent $+1, -5, +10, -10, +5, -1$, vel $+1, +5, +10, -10, -5, -1$, ubi Ordinatæ a medio æqualiter remotæ ſunt etiamnum æquales, at diverſis ſignis affectæ; in eo inquam caſu defuiſſent dignitates Abſciſſæ parium dimenſionum.

Scholion.

De deſcriptione Curvæ Parabolici generis per data quotcunque puncta, egerunt plures celebres Geometræ poſt *Newtonum*. Sed omnes eorum ſolutiones eædem ſunt cum hiſce jam exhibitis; quæ quidem a *Newtonianis* vix diſcrepant, uti conſtabit ex Lemmate quinto Libri tertii *Principiorum* & *methodo Differentiali* a D. *Jones* edita. *Newtonus* quidem deſcribit Parabolam per data puncta; alii conſideraverunt aſſignationem Terminorum ex datis differentiis; ſed quocunque modo concipiatur, ſub quâcunque formâ proferatur, idem eſt Problema. Et ſane

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ſane inventio formarum quas habent valores Ordinatæ T, eſt perquam ingenioſa & ſummo Auctore digna: at poſtquam habentur formæ, inveſtigatio Problematis eſt facilis, in quam utique nihil aliud requiritur quam reſolutio Æquationum ſimplicium.

Sed notandum eſt formam Ordinatæ $A+Bz+Cz^2+Dz^3+$ &c. ex poteſtatibus compoſitæ, quam aſſumit *Newtonus* in demonſtrando ſuæ methodi fundamento, male huic negotio deſtinatam eſſe. Nam valor cujuſque Coefficientis prodit in Serie infinita; ſi quis vero aſſumſerit formas hic uſurpatas, incidet minimo labore in concluſiones ſuperiores.

Propositio XXI.

Datâ Serie Terminorum primariorum invenire intermedios, qui non longe diſtant a principio.

Super rectam poſitione datam ad anglos rectos, & in æqualibus ab invicem diſtantiis erigantur Ordinatæ reſpective æquales Terminis primariis; dein per Propoſitiones duas ſuperiores quæratur Linea Parabolica quæ tranſit per omnium extremitates, & eadem quoque tranſibit per extremitates intermediorum; qui itaque dabuntur ex data Æquatione ad Parabolam. Q. E. I.

Exemplum I.

Interpolanda ſit Series $1, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{35}{128}, \frac{63}{256},$ &c. cujus Termini producuntur per multiplicationem continuam numerorum $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8},$ &c. quære differentias Terminorum & differentias differentiarum, ut ſequitur.

$$\begin{array}{ccccccccccc}
1 & & \frac{1}{2} & & \frac{3}{8} & & \frac{5}{16} & & \frac{35}{128} & & \frac{63}{256}\ \&c. \\
 & -\frac{1}{2} & & -\frac{1}{8} & & -\frac{1}{16} & & -\frac{5}{128} & & -\frac{7}{256} & \\
 & & \frac{3}{8} & & \frac{1}{16} & & \frac{3}{128} & & \frac{3}{256} & & \\
 & & & -\frac{5}{16} & & -\frac{5}{128} & & -\frac{7}{256} & & & \\
 & & & & \frac{35}{128} & & \frac{7}{256} & & & & \\
 & & & & & -\frac{63}{256} & & & & &
\end{array}$$

Quoniam

Quoniam hæc Series ex unâ tantum parte excurrit in infinitum, peragenda eſt Interpolatio per Propoſitionem decimam nonam. Et ſi primus Terminus 1 adhibeatur pro Ordinata prima, erit $A = 1$, $B = -\frac{1}{2}$, $C = +\frac{3}{8}$, $D = -\frac{5}{16}$, $E = +\frac{35}{128}$, &c. quibus ſubſtitutis, prodit

$$T = 1 - \frac{1}{2} \times \frac{z}{1} + \frac{3}{8} \times \frac{z}{1} \times \frac{z-1}{2} - \frac{5}{16} \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} + \&c.$$

id eſt, $T = 1 - \frac{1}{2} A \frac{z}{1} - \frac{3}{4} B \frac{z-1}{2} - \frac{5}{6} C \frac{z-2}{3} - \frac{7}{8} D \frac{z-3}{4} - \&c.$

Ubi jam A, B, C, D, &c. deſignant Terminos hujus Seriei, more *Newtoni*. Ceterum Terminus quilibet primarius uſurpari poteſt pro Ordinata prima, eritque $A = \frac{1}{2}$, $B = -\frac{1}{8}$, $C = +\frac{1}{16}$, $D = -\frac{5}{128}$, $E = +\frac{7}{256}$, &c. adeoque

$$T = \frac{1}{2} - \frac{1}{8} \times \frac{z}{1} + \frac{1}{16} \times \frac{z}{1} \times \frac{z-1}{2} - \&c.$$

id eſt, $T = \frac{1}{2} - \frac{1}{4} A \frac{z}{1} - \frac{3}{6} B \frac{z-1}{2} - \frac{5}{8} C \frac{z-2}{3} - \frac{7}{10} D \frac{z-3}{4} - \&c.$

Sed ſciendum eſt z eſſe diſtantiam inter Terminum quæſitum & illum qui adhibetur pro Ordinata prima. Ut ſi Terminus deſideratus T ſtet in medio inter primum & ſecundum, pone $+\frac{1}{2}$ pro z in priore valore ipſius T; & $-\frac{1}{2}$ pro eadem in poſteriore valore; atque pro eodem Termino habebis duas Series ſequentes

$$1 - \frac{1}{4} A + \frac{3}{16} B + \frac{15}{36} C + \frac{35}{64} D + \frac{63}{100} E + \&c.$$

$$\frac{1}{2} + \frac{1}{8} A + \frac{9}{24} B + \frac{25}{48} C + \frac{49}{80} D + \frac{81}{120} E + \&c.$$

Et quoniam hæ Series convergunt quam lentiſſime, ſummandæ ſunt per Theorema in Scholio Propoſitionis decimæ primæ. Atque ſciendum eſt valores Terminorum prodire ſimpliciſſimos quando Terminus qui conſiſtit proximus intermedio quæſito, adhibetur pro Ordinata prima. Sed quando Terminus quæſitus longiſſime diſtat ab initio, uſui venit Propoſitio decima octava, ut monſtrabitur in ſequentibus.

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Exemplum II.

Interpolanda ſit Series 1, 1, 2, 6, 24, 120, 720, &c. cujus Termini generantur per multiplicationem continuam numerorum 1, 2, 3, 4, 5, &c. Quoniam hi Termini increſcunt celerrime, eorum differentiæ conficient progreſſionem divergentem, id quod impedit quo minus Ordinata Parabolæ accedat ad veritatem. In hoc igitur & ſimilibus caſibus interpolo Logarithmos Terminorum, quorum utique differentiæ conſtituere queunt Seriem celeriter convergentem, etiamſi ipſi Termini increſcunt celerrime ut in Exemplo præſente.

Propono jam inventionem Termini qui conſiſtit in medio inter duos primos 1 & 1. Et quoniam Logarithmi Terminorum initialium habent differentias lente convergentes, primum quæram Terminum in medio conſiſtentem inter duos ab initio ſatis remotos, verbi gratiâ, inter decimum primum 3628800 & decimum ſecundum 39916800: & ex eo dato regrediar ad Terminum quæſitum per Propoſitionem decimam ſextam. Quumque habeantur aliquot Termini conſiſtentes ex utrâque parte intermedii qui primo eruendus eſt, inſtituo operationem per Caſum ſecundum Propoſitionis vigeſimæ. Etenim ubi computus non eſt in ſpeciebus, ſed in numeris, procedere licet hâc methodo quotieſcunque datur ſatis amplus Terminorum numerus ex utraque parte Termini quæſiti conſiſtentium, etiamſi ipſa Series interpolanda reapſe non excurrat utrinque in infinitum.

Excerpo nunc ex Tabula Logarithmos duodecim Terminorum, quorum primus eſt is Termini ſexti 120; ſic ut habeantur ſex ante, totidemque poſt illum qui quæritur. Dein quoniam Terminus ille deſideratus conſiſtit directe in medio omnium, erit in Caſu ſecundo Propoſitionis vigeſimæ primæ Abſciſſa $z = 0$; & propterea differentiæ primæ, tertiæ, reliquæque ordinum imparium quæ in z multiplicantur, non ingredientur computum; igitur tantummodo colligo differentias ſecundas, quartas, ceteraſque in ordinibus paribus ut vides

Logarithmi

Logarithmi	Diffe. 2dæ	quartæ	ſextæ	octavæ	decimæ
2.0791812460					
2.8573324964	669467896				
3.7024305364	579919470	21154180			
4.6055205234	511525224	14443928	2568588		
5.5597630329	457574906	10302264	1446210	541511	
6.5597630329	413926852	7606810	865343	259252	156590
7.6011557180	377885608	5776699	543728	133583	65082
8.6803369641	347621063	4490316	355696	72996	
9.7942803164	321846834	3559629	240660		
10.9404083521	299632234	2869602			
12.1164996111	280287236				
13.3206195938					

Excerpo jam binos medios Logarithmos, binaſque medias differentias; eorumque Summas pono æquales A, B, C, D, &c. reſpective ut vides

$$\begin{array}{r} 6.5597630329 \\ 7.6011557180 \\ \hline A = 14.1609187509, \end{array} \qquad \begin{array}{r} 413926852 \\ 377885608 \\ \hline B = 791812460, \end{array} \qquad \begin{array}{r} 7606810 \\ 5776699 \\ \hline C = 13383509, \end{array}$$

$$\begin{array}{r} 865343 \\ 543728 \\ \hline D = 1409071, \end{array} \qquad \begin{array}{r} 259252 \\ 133583 \\ \hline E = 392835, \end{array} \qquad \begin{array}{r} 156590 \\ 65082 \\ \hline F = 221672, \end{array}$$

Tum in Caſu ſecundo Propoſitionis vigeſimæ primæ; ſubſtituo o pro z, & habeo

$$T = \frac{1}{2} A - \frac{1}{16} B + \frac{3}{256} C - \frac{5}{2048} D + \frac{35}{65536} E - \frac{63}{524288} F + \&c.$$

Scribe jam ipſorum A, B, C, D, E, F, valores mox repertos, & computus ſic ſe habebit

$$\begin{array}{r} 7.08045937545 \\ 1568380 \\ 2098 \\ \hline + 7.08047508023 \end{array} \qquad \begin{array}{r} 494882787 \\ 34401 \\ 266 \\ \hline - 494917454 \end{array}$$

Ac ſubducendo Summam Terminorum negativorum ab eâ affirmativorum, remanet $T = 7.07552590569$. Eſtque hic Logarithmus numeri 11899423.08, utique qui conſiſtit in medio inter Terminos 3628800 & 39916800.

Jam

Jam ſicuti Termini primarii formantur ducendo primum continue in numeros 1, 2, 3, 4, &c. ita per Propoſitionem decimam ſextam, intermedii generantur ducendo intermedium inter primum & ſecundum in numeros $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, &c. continue. Verbi gratiâ productum ſub decem factoribus $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, $5\frac{1}{2}$, $6\frac{1}{2}$, $7\frac{1}{2}$, $8\frac{1}{2}$, $9\frac{1}{2}$, $10\frac{1}{2}$, & Termino qui conſiſtit in medio inter primum & ſecundum, æquale eſt intermedio mox invento 11899423.08, cujus utique locus eſt in medio inter decimum primum & decimum ſecundum. Igitur ſi Terminus ille intermedius dividatur per $10\frac{1}{2}$, & quotus per $9\frac{1}{2}$, & quotus noviſſimus per $8\frac{1}{2}$, & ſic porro uſque ad diviſorem $1\frac{1}{2}$; ultimus quotorum æqualis erit Termino in medio inter 1 & 1. Termini vero intermedii per diviſionem illam prodeuntes ſunt

Primarii	Intermedii
39916800	11899423.08
3628800	1133278.389
362880	119292.4620
40320	14034.40729
5040	1871.254305
720	287.8852777
120	52.34277777
24	11.63172839
6	3.323350969
2	1.329340388
1	.8862269251
1	1.77245385O2

Unde conſtat Terminum inter 1 & 1 eſſe .8862269251; cujus quadratum eſt .7853..., &c. ſcilicet Area Circuli cujus Diameter eſt unitas. Atque illius Termini duplus 1.7724538502, ſcilicet qui conſiſtit ante primum primarium dimidio intervalli communis, æqualis eſt Radici quadrati numeri 3.1415926.... &c. qui denotat Circumferentiam Circuli

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Circuli, cujus Diameter eſt unitas. Etenim ſi quadrata primariorum conſtituant novam Seriem 1, 1, 4, 36, 576, 14400, &c. Terminus in medio inter primum & ſecundum erit ad unitatem ut Area Circuli ad quadratum circumſcriptum: & Terminus qui conſiſtit ante primum dimidio intervalli communis erit ad unitatem ut Circumferentia Circuli ad Diametrum. Ceterum in ſequentibus monſtrabitur hujuſmodi Series interpolari poſſe abſque Logarithmis.

Exemplum III.

Quadranda ſit Curva cujus Ordinata eſt $x^{\theta-1} \times \overline{e+fx^{\eta}}^{\lambda}$. Scribe 0, 1, 2, 3, &c. ſucceſſive pro indice λ, & proveniet Series Ordinatarum æquidiſtantium $x^{\theta-} \times \overline{e+fx^{\eta}}^{0}$, $x^{\theta-1} \times \overline{e+fx^{\eta}}^{1}$, $x^{\theta-1} \times \overline{e+fx^{\eta}}^{2}$, &c. inter quas Ordinata propoſita locum obtinet intervallo λ diſtantem ab initio. Igitur Area quæſita eundem etiam obtinebit locum inter Areas illarum Ordinatarum, quæ conſtituunt progreſſionem ſequentem Terminorum æquidiſtantium,

$$\frac{1}{\theta}x^{\theta},\ \frac{e}{\theta}x^{\theta}+\frac{f}{\theta+\eta}x^{\theta+\eta},\ \frac{e^2}{\theta}x^{\theta}+\frac{2ef}{\theta+\eta}x^{\theta+\eta}+\frac{f^2}{\theta+2\eta}x^{\theta+2\eta},\ \&c.$$

Jam ſi hæ Areæ interpolarentur per Propoſitionem decimam nonam, proveniret pro Areâ quæſitâ, eadem Series ac per methodum vulgarem reducendi Ordinatam in Seriem convergentem ut exinde inveniatur fluens. Attamen ſi Termini in Serie Arearum primo dividantur per Terminos hujus progreſſionis Geometricæ reſpective, nempe $\overline{e+fx^{\eta}}^{0}$, $\overline{e+fx^{\eta}}^{1}$, $\overline{e+fx^{\eta}}^{2}$, $\overline{e+fx^{\eta}}^{3}$, &c. Hoc eſt, ſi ponantur

$$A = x^{\theta} \times \frac{\frac{1}{\theta}}{\overline{e+fx^{\eta}}^{0}},$$

$$A1 = x^{\theta} \times \frac{\frac{e}{\theta}+\frac{fx^{\eta}}{\theta+\eta}}{\overline{e+fx^{\eta}}^{1}},$$

$$A2 = x^{\theta} \times \frac{\frac{e^2}{\theta}+\frac{2efx^{\eta}}{\theta+\eta}+\frac{f^2x^{2\eta}}{\theta+2\eta}}{\overline{e+fx^{\eta}}^{2}},$$

$$A3 = x^{\theta} \times \frac{\frac{e^3}{\theta}+\frac{3e^2fx^{\eta}}{\theta+\eta}+\frac{3ef^2x^{2\eta}}{\theta+2\eta}+\frac{f^3x^{3\eta}}{\theta+3\eta}}{\overline{e+fx^{\eta}}^{3}},$$

&c.

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& inſtituatur computus juxta Propoſitionem decimam nonam, invenientur differentiæ

$$\mathrm{B} = \frac{-nfx^{\theta+n}}{\theta.\overline{\theta+n}.\overline{e+fx^n}},\ \mathrm{C} = \frac{+2n^2f^2x^{\theta+2n}}{\theta.\overline{\theta+n}.\overline{\theta+2n}.\overline{e+fx^n}^2},\ \mathrm{D} = \frac{-6n^3f^3x^{\theta+3n}}{\theta.\overline{\theta+n}.\overline{\theta+2n}.\overline{\theta+3n}.\overline{e+fx^n}^3},\ \&c.$$

quibus ſubſtitutis pro A, B, C, D, &c. & λ pro z; invenies Terminum diſtantiâ λ ab initio remotum, eſſe

$$\frac{x^\theta}{\theta} - \frac{\lambda nfx^{\theta+n}}{\theta.\overline{\theta+n}.\overline{e+fx^n}} + \frac{\lambda.\overline{\lambda-1}.n^2f^2x^{\theta+2n}}{\theta.\overline{\theta+n}.\overline{\theta+2n}.\overline{e+fx^n}^2} - \frac{\lambda.\overline{\lambda-1}.\overline{\lambda-2}.n^3f^3x^{\theta+3n}}{\theta.\overline{\theta+n}.\overline{\theta+2n}.\overline{\theta+3n}.\overline{e+fx^n}^3} + \&c.$$

Sed quia Termini interpolandi diviſi erant per dignitates ipſius $e+fx^n$, quiſque ſcilicet per dignitatem cujus index erat diſtantia Termini ab initio; e contra duc Terminum mox inventum in dignitatem prædicti binomii, cujus index eſt λ, ſua utique diſtantia ab initio, & pro Area Curvæ habebitur

$$\frac{x^\theta}{\theta} \times \overline{e+fx^n}^{\lambda} \text{ in } 1 - \frac{\lambda nfx^n}{\theta+n.\overline{e+fx^n}} + \frac{\lambda.\overline{\lambda-1}.n^2f^2x^{2n}}{\theta+n.\theta+2n.\overline{e+fx^n}^2} - \&c.$$

Vel ponendo $y = \frac{fx^n}{e+fx^n}$, $r = \frac{\theta+n}{n}$; atque ſcribendo Seriem more *Newtoni*, prodibit

$$\frac{x^\theta}{\theta} \times \overline{e+fx^n}^{\lambda} - \frac{\lambda}{r}\mathrm{A}y - \frac{\lambda-1}{r+1}\mathrm{B}y - \frac{\lambda-2}{r+2}\mathrm{C}y - \frac{\lambda-3}{r+3}\mathrm{D}y - \&c.$$

pro Area Curvæ cujus Ordinata in genere eſt $x^{\theta-1} \times \overline{e+fx^n}^{\lambda}$. Et hæc Series tranſmutata per Propoſitionem ſeptimam, migrabit in eam *Newtoni* pro quadraturâ Curvæ binomialis. Abrumpit quando index λ eſt integer & affirmativus; ſed poſt debitam Ordinatæ præparationem abrumpet ſemper ubi Curva eſt quadrabilis. Ceterum ejus uſus præcipuus eſt quod exhibeat Areas in Serie admodum ſimplici. Si Coefficientes e, f, contrariis ſignis afficiantur, præferenda eſt Series *Newtoni*; & noſtra ubi iiſdem.

EXEMPLUM IV.

Oporteat aſſignare Uncias Binomii ex data Uncia media in dignitate cujus index eſt numerus par. Si n denotet indicem dignitatis, & Uncia media ducatur continue in fractiones $\frac{n}{n+2}$, $\frac{n-2}{n+4}$, $\frac{n-4}{n+6}$, &c. facta erunt Unciæ reliquæ conſiſtentes ex utraque parte mediæ.

$a6 =$

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$$a6 = a \times \frac{n}{n+2} \times \frac{n-2}{n+4} \times \frac{n-4}{n+6},$$

$$a4 = a \times \frac{n}{n+2} \times \frac{n-2}{n+4},$$

$$a2 = a \times \frac{n}{n+2},$$

$$a = a,$$

$$2a = a \times \frac{n}{n+2},$$

$$4a = a \times \frac{n}{n+2} \times \frac{n-2}{n+4},$$

$$6a = a \times \frac{n}{n+2} \times \frac{n-2}{n+4} \times \frac{n-4}{n+6},$$

Et ſi a deſignet Unciam mediam, tum $a2$, $a4$, $a6$, &c. ex una parte; & $2a$, $4a$, $6a$, &c. ex alterâ deſignabunt reliquas Uncias. Dein ineundo computum per caſum primum Propoſitionis vigeſimæ invenies

$$a - \frac{rr}{2.\overline{n+2}} A - \frac{rr-4}{4.\overline{n+4}} B - \frac{rr-16}{6.\overline{n+6}} C - \frac{rr-36}{8.\overline{n+8}} D - \&c.$$

eſſe Terminum Seriei interpolandæ, cujus diſtantia a Termino media a eſt ad intervallum commune primariorum ut r ad Binarium. Exempli gratiâ, in dignitate decimâ ſecundâ Unciæ ſunt 1, 12, 66, 220, 495, 792, 924, 792, &c. exiſtente Unciâ mediâ $a = 924$. Et ſi quæratur ea quæ ternario diſtat a media, erit $r = 6$; quo ſubſtituto, & 12 pro n; prodit

$$924 - \frac{36}{2.14} A - \frac{32}{4.16} B - \frac{20}{6.18} C,$$

pro Uncia quæſita, abrumpente Serie: & hi Termini liberati a fractionibus ſunt $925 - 1188 + 594 - 110$, quorum ſumma ſub propriis ſignis eſt 220, qui eſt valor Unciæ quæſitæ.

Et ad eundem modum ſi index n ſit numerus impar, & A altera e mediis Unciis; tum erit

$$A - \frac{rr-1}{2.\overline{n+3}} A - \frac{rr-9}{4.\overline{n+5}} B - \frac{rr-25}{6.\overline{n+7}} C - \frac{rr-49}{8.\overline{n+9}} D - \&c.$$

Uncia cujus diſtantia a puncto inter duas medias Uncias intermedio, eſt ad intervallum commune ut r ad Binarium.

Et hæ Series in permagna dignitate convergent dummodo intervallum inter Unciam mediam & quæſitam ſit exiguum reſpectu indicis poteſtatis.

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Scholion.

Poſtquam Series interpolanda debite præparatur per Propoſitionem decimam ſeptimam; etiamſi eadem utrinque excurrat in infinitum procedere licet per Propoſitionem decimam nonam præterquam ubi Termini a medio hinc inde æqualiter diſtantes ſunt inter ſe æquales: ubi hoc accidit, adhibeatur caſus primus Propoſitionis vigeſimæ, ſi Terminus quivis primarius quodam jure vindicet locum in medio omnium: vel ſi duo Termini eodem jure locum medium vindicent, adhibeatur caſus ſecundus ejuſdem Propoſitionis. Et in aliis caſibus ad libitum fere proceditur.

Propositio XXII.

Datâ Serie Terminorum æquidiſtantium, invenire Terminum quemvis primarium vel intermedium utcunque longe diſtantem ab initio Seriei.

Si Terminus quæſitus longe removetur a principio, tum per Propoſitionem decimam octavam quære aliam Seriem in qua Terminus ille deſideratus conſtituet Terminum prope initium, dein procede ut in Propoſitione ſuperiore.

Exemplum I.

Proponatur invenire Terminum quemvis hujus Seriei, $1, \frac{2}{1}A, \frac{4}{3}B, \frac{6}{5}C, \frac{8}{7}D$, &c. a principio quantumvis magno intervallo m diſtantem. Per Propoſitionem decimam octavam, Terminus Seriei

$$1, \frac{a}{m+1}, \frac{2b}{m+2}, \frac{3c}{m+3}, \frac{4d}{m+4}, \&c.$$

qui conſiſtit ante primum dimidio intervalli communis, æqualis erit Termino Seriei prioris cujus intervallum ab initio eſt m. Conſtat autem per Exemplum ſecundum Propoſitionis vigeſimæ primæ, Terminum dimidio intervalli communis diſtantem ante primum in Serie Numeratorum, $1, 1.1, 1.1.2, 1.1.2.3$, &c. id eſt, in hac $1, 1, 2, 6, 24, 120$, &c. eſſe Radicem quadratam Numeri $3.1415926 \ldots$ &c. Quare tantum interpolo Denominatores nempe

$$1, \frac{1}{m+1}, \frac{1}{m+1.m+2}, \frac{1}{m+1.m+2.m+3}, \&c.$$

Et

Et quoniam hæc Series continuari poteſt utrinque in infinitum; actu continuabitur, & evadet

$$\&c.\ \overline{m-2.m-1}.m,\ \overline{m-1}.m,\ m,\ ,1,\ \frac{1}{m+1},\ \frac{1}{\overline{m+1.m+2}},\ \&c.$$

Ubi Terminus quæſitus conſiſtit in medio inter duos medios m & 1: ſed quia differentiæ horum Terminorum ſunt permagnæ; ducantur ii a medio hinc inde æqualiter diſtantes in ſe mutuo; hoc eſt, m in 1; $\overline{m-1}.m$ in $\frac{1}{m+1}$; & ſic porro; & prodibit nova Series

$$\&c.\ \frac{m-2}{m+2}\cdot\frac{m-1}{m+1}\cdot m,\ \frac{m-1}{m+1}m,\ m,\ m,\ m\frac{m-1}{m+1},\ m\frac{m-1}{m+1}\cdot\frac{m-2}{m+2},\ \&c.$$

utrinque excurrens in infinitum, & habens Terminos inter ſe æquales qui æqualiter diſtant a medio. Sed & Terminus in medio conſiſtens inter duos medios primarios m & m eſt quadratum ejus in medio inter m & 1 in Serie priore. In noviſſimâ igitur Serie quæratur Terminus ille inter m & m per Caſum ſecundum Propoſitionis vigeſimæ, & idem invenietur

$$m+\frac{m}{4.\overline{m+1}}+\frac{9m}{32.\overline{m+1.m+2}}+\frac{75m}{128.\overline{m+1.m+2.m+3}}+\&c.$$

Quam ducito in Circumferentiam Circuli, ſcilicet Terminum reſpectivum in Serie Numeratorum, & habebis pro quadrato Termini quæſiti

$$3.14159\ldots\ \&c.\ \text{in}\ m+\frac{A}{4.\overline{m+1}}+\frac{9B}{8.\overline{m+2}}+\frac{25C}{12.\overline{m+3}}+\frac{49D}{16.\overline{m+4}}+\&c.$$

Igitur Terminus Seriei propoſitæ $1,\ \frac{2}{1}A,\ \frac{4}{3}B,\ \frac{6}{5}C$, &c. diſtans intervallo m ab initio æqualis eſt mediæ proportionali inter Circumferentiam Circuli & Seriem illam; quæ eo citius convergit quo major eſt m, hoc eſt quo longius diſtat ab initio Terminus quæſitus.

Exempli gratiâ ſit $n=100$ & primus Terminus Seriei erit Circumferentia Circuli ducta in 100, ſive $A=314.15927$; dein erit $B=\frac{A}{4.101}=.77762$, $C=\frac{9B}{8.102}=.00858$, $D=\frac{25C}{12.103}=.00017$; atque ſumma horum quatuor Terminorum eſt 314.94564, cujus Radix quadrata 17.7467079 eſt Terminus centeſimus primus Seriei interpolandæ; ſive productum ſub factoribus $\frac{2}{1}\times\frac{4}{3}\times\frac{6}{5}\times\frac{8}{7}$, &c. quorum numerus eſt Centum. Et eâdem ratione invenire licet Terminum quemvis inter-

$$\begin{array}{r} 314.15927 \\ 77762 \\ 858 \\ 17 \\ \hline 314.94564 \end{array}$$

 medium:

medium: nam ſi pro m ſcribatur $99\frac{1}{2}$, habebitur Terminus in medio inter centeſimum & centeſimum primum. Vel ſi ſubſtituatur $99\frac{1}{3}$ pro m, habebitur Terminus conſiſtens poſt centeſimum tertiâ parte intervalli communis.

Notandum eſt, poſſe Terminorum alicujus Seriei reciprocos interpolari: ſic reciproci Terminorum in Serie noviſſimâ conſtituunt Seriem $1, \frac{1}{2}A, \frac{3}{4}B, \frac{5}{6}C, \frac{7}{8}D$, &c. in qua Terminus ab initio remotus intervallo quovis æquali $\frac{1}{2}m$, æqualis erit Termino Seriei

$$1, \frac{a}{m+1}, \frac{3b}{m+3}, \frac{5c}{m+5}, \frac{7d}{m+7}, \text{ \&c.}$$

qui conſiſtit in medio inter primum & ſecundum; quique proinde invenietur eſſe media proportionalis inter Seriem ſequentem

$$\frac{1}{m+1} + \frac{A}{2.\overline{m+3}} + \frac{9B}{4.\overline{m+5}} + \frac{25C}{6.\overline{m+7}} + \frac{49D}{8.\overline{m+9}} + \text{\&c.}$$

& numerum .6366197723676 qui æqualis eſt unitati diviſæ per ſemicircumferentiam Circuli: id quod conſtabit inſiſtendo veſtigiis prioris partis hujus Exempli.

Exemplum II.

Quæratur Terminus hujus Seriei $1, \frac{2}{1}A, \frac{5}{4}B, \frac{8}{7}C, \frac{11}{10}D$, &c. quantumvis remotus ab initio; ſcilicet intervallo m: & per Propoſitionem decimam octavam Terminus ille æquabitur Termino hujus Seriei

$$1, \frac{2a}{3m+2}, \frac{5b}{3m+5}, \frac{8c}{3m+8}, \frac{11d}{3m+11}, \text{ \&c.}$$

qui conſiſtit ante primum tertia parte intervalli communis. Igitur intercalentur Numeratores & Denominatores ſeorſim ut in Exemplo ſecundo Propoſitionis vigeſimæ primæ, ſcilicet per Logarithmos; & habebitur Terminus quæſitus.

Scholion.

Hinc conſtat Terminos Serierum longiſſime diſtantes erui poſſe non minus accurate ac intermedii ſub initio. Sed in Serie interpolandâ 1,

$\frac{r}{p}$

$\frac{r}{p}$ A, $\frac{r+1}{p+1}$ B, &c. ſi differentia inter p & r ſit magna, magnus item erit labor in inveniendo quovis Termino. Caſus autem omnium facillimus eſt ubi $p-r$ eſt æqualis $\pm\frac{1}{2}$, ut in Exemplo primo, præterquam ubi differentia illa eſt numerus integer quo in Caſu Series erit accurate interpolabilis.

Propositio XXIII.

Invenire rationem quam habet Uncia media ad Summam omnium Unciarum in quacunque dignitate binomii.

Solutio prior.

Si index dignitatis ſit numerus par, appelletur n; vel ſi ſit impar, vocetur $n-1$; eritque ut Uncia media ad Summam omnium ejuſdem dignitatis, ita Unitas ad mediam proportionalem inter ſemicircumferentiam Circuli & alterutram Serierum ſequentium

$$n+\frac{A}{2.\overline{n+2}}+\frac{9B}{4.\overline{n+4}}+\frac{25C}{6.\overline{n+6}}+\frac{49D}{8.\overline{n+8}}+\frac{81E}{10.\overline{n+10}}+\&c.$$

Vel

$$n+1-\frac{A}{2.\overline{n-3}}-\frac{9B}{4.\overline{n-5}}-\frac{25C}{6.\overline{n-7}}-\frac{49D}{3.\overline{n-9}}-\frac{81E}{10.\overline{n-11}}-\&c.$$

Exempli gratiâ, ſi quæratur ratio Unciæ mediæ ad Summam omnium in dignitate centeſimâ vel nonageſimâ nonâ, erit $n=100$; qui multiplicatus in ſemiperiferiam Circuli, producit primum Terminum $A=157.0796326 79$; dein erit $B=\frac{A}{204}$, $C=\frac{9B}{416}$, $D=\frac{25C}{636}$, &c. atque perficiendo computum ut in margine, invenietur Summa Terminorum 157.866984459, cujus Radix quadrata 12.5645129018 eſt ad unitatem ut Summa omnium Unciarum ad mediam in dignitate centeſimâ vel nonageſimâ nonâ. Eſt autem hic computus per priorem Seriem: nam licet exigua ſit differentia, præfero eam in quâ Termini ſunt omnes ejuſdem ſigni.

157.079632679
769998199
166586615
654820
37137
2734
246
26
3
157.866984459

Solutio

Solutio poſterior.

Manente n ut antea; erit Summa omnium Unciarum ad mediam in ſubduplicatâ ratione periferiæ Circuli ad alterutram Serierum ſequentium,

$$\frac{1}{n+1} + \frac{A}{2.\overline{n+3}} + \frac{9B}{4.\overline{n+5}} + \frac{25C}{6.\overline{n+7}} + \frac{49D}{8.\overline{n+9}} + \frac{81E}{10.\overline{n+11}} + \&c.$$

Vel

$$\frac{1}{n} - \frac{A}{2.\overline{n-2}} - \frac{9B}{4.\overline{n-4}} + \frac{25C}{6.\overline{n-6}} + \frac{49D}{8.\overline{n-8}} + \frac{81E}{10.\overline{n-10}} + \&c.$$

Vel quod eodem redit, ponatur $a = .6366197723676$, quoto ſcilicet qui prodit dividendo unitatem per ſemicircumferentiam Circuli; & media proportionalis inter numerum a & alterutram illarum Serierum erit ad unitatem ut Uncia media ad ſummam omnium.

Ut ſi ſit index $n = 100$, ut in computo ſuperiore; erit per Seriem primam $A = \frac{a}{101}$, $B = \frac{A}{206}$, $C = \frac{9B}{420}$, $D = \frac{25C}{642}$, &c. Ex calculo autem appoſito patet Summam Terminorum eſſe .00633444670787, cujus Radix quadrata .0795892373872 eſt ad unitatem ut Uncia media ad Summam omnium in dignitate nonageſimâ nonâ vel centeſimâ.

.00630316606305
3059789351
65566914
2553229
143473
10469
934
98
12
2
.00633444670787

Et ſic omnino ſunt quatuor Series ejuſdem ſimplicitatis pro ſolutione hujus Problematis. Ceterum in praxi non opus eſt recurrere ad Series: nam ſufficit ſumere mediam proportionalem inter ſemicircumferentiam Circuli & $n + \frac{1}{2}$; hæc enim ſemper approximabit propius quam duo primi Termini Serici, quorum etiam primus ſolus plerumque ſufficit. Exempli gratiâ, ſi ſit $n = 100$, erit $n + \frac{1}{2} = 100\frac{1}{2}$, qui ductus in ſemicircumferentiam Circuli producit 157.865, cujus Radix quadrata eſt 12.5644, unitate deficiens in ultimâ figurâ.

Eadem vero approximatio aliter & praxi forte commodior ſic enunciatur. Pone c ad unitatem ut quadratum Diametri ad Circulum; hoc eſt, ſit $c = 1.2732395447352$; eritque ſumma Unciarum ad mediam ut Uncia

Unitas ad $\sqrt{\frac{c}{2n+1}}$ quamproxime, exiſtente errore in exceſſu circiter $\frac{1}{16nn}\sqrt{\frac{c}{2n+1}}$. Sit $n=100$, erit $\frac{c}{2n+1}=.006334525$, ejuſque Radix quadrata .07958973 accurata eſt in ſextâ decimali: quæ vero diviſa per $16nn$, hoc eſt per 160000, dabit correctionem .00000050; & hæc ſubducta de Approximatione, relinquit numerum quæſitum .07958923 accuratum in ultima figurâ.

Similiter ſit $n=900$, erit $\frac{c}{2n+1}=.000706962545$, cujus Radix quadrata .026588767 ſuperat verum binario in ultimâ figurâ. Sin vero correctio computetur ac ſubducatur de Approximatione, habebitur numerus quæſitus accuratus in decimâ tertia decimali.

Hem vero Approximationem æque facilem & magis accuratam. Differentia inter Logarithmos numerorum $n+2$ & $n-2$, dividatur per 16, & quotus adjiciatur dimidio Logarithmi indicis n, huic dein ſummæ addatur Logarithmus conſtans .0980599385151, ſcilicet dimidium Logarithmi ſemicircumferentiæ Circuli, & ſumma noviſſima erit Logarithmus numeri qui eſt ad unitatem ut ſumma Unciarum ad mediam. Verbi gratiâ, ſit $n=900$, computus erit

$\frac{1}{2}$ *l*, 900	1.4771212547
16) Diff. Log. 902 & 898	.0001206376
Log. conſtans	.0980599385
Summa	1.5753018308

Et hæc Summa verum ſuperat binario, in ultimâ figurâ, eſtque Logarithmus numeri 37.6098698 qui eſt ad unitatem ut ſumma Unciarum ad mediam in poteſtate 900 vel 899. Et ſi vis illius numeri reciprocum, ſume complementum Logarithmi, ſcilicet —2.4246981692, & numerus eidem correſpondens erit .0265887652 qui monſtrat rationem Unciæ mediæ ad ſummam omnium in prædictis dignitatibus.

Demonstratio.

Dignitates Binomii, quarum indices ſunt numeri pares, habent unam Unciam mediam; eæ vero, quarum indices ſunt impares, habent duas Uncias medias. Et hinc oriuntur duo Caſus Problematis. Primo quando index eſt par, divide ſummas Unciarum 1, 4, 16, 64, 256, 1024, &c. per ſuas Uncias medias, 1, 2, 6, 20, 70, 252, &c. & quoti 1, 2, $\frac{8}{3}$, $\frac{16}{5}$, $\frac{128}{35}$, $\frac{256}{63}$, &c. ſive 1, $\frac{2}{1}$ A, $\frac{4}{3}$ B, $\frac{6}{5}$ C, $\frac{8}{7}$ D, &c. e-

 runt

runt ad unitatem ut ſummæ Unciarum ad Uncias medias in diverſis dignitatibus.

Similiter ſi ſummæ Unciarum in dignitatibus imparibus, ſcilicet 2, 8, 32, 128, 512, &c. dividantur per Uncias ſuas medias 1, 3, 10, 35, 126, &c. rurſus prodibunt idem quoti, utpote 2, $\frac{8}{3}$, $\frac{16}{5}$, $\frac{128}{35}$, &c. Nam eadem eſt relatio inter ſummam Unciarum & Unciam mediam in quavis dignitate pare, quæ eſt inter ſummam Unciarum & Unciam mediam in dignitate impare proxime inferiori. Adeoque interpolatio Seriei, 1, $\frac{2}{1}$ A, $\frac{4}{3}$ B, $\frac{6}{5}$ C, $\frac{8}{7}$ D, &c. ut in Exemplo primo Propoſitionis vigeſimæ ſecundæ, ſolvit utrumque Caſum Problematis. Sed hic dabimus inveſtigationem harum Serierum abſque methodo Differentiali.

Analyſis Solutionis prioris.

Series interpolanda 1, $\frac{2}{1}$ A, $\frac{4}{3}$ B, &c. definitur Æquatione $T' = \frac{n+2}{n+1} T$, ubi n eſt quantitas variabilis, ejuſque valores ſucceſſivi 0, 2, 46, 8, &c. ſcilicet indices dignitatum quando ſunt pares, vel indices aucti unitate quando ſunt impares. Quadra utramque partem Æquationis reſolvendæ, & habebitur $T'T' = \frac{nn+4n+4}{nn+2n+1} TT$ vel quod idem eſt $2T'T' + \overline{n+2} \times \overline{TT - T'T'} - \frac{T'T'}{n+2} = 0$. Aſſume jam

$$TT = An + \frac{Bn}{n+2} + \frac{Cn}{n+2.n+4} + \frac{Dn}{n+2.n+4.n+6} + \&c.$$

Et poſt debitam reductionem juxta præcepta jam tradita, invenies

$$TT = An + B + \frac{C-2B}{n+2} + \frac{D-4C}{n+2.n+4} + \&c.$$

In quâ ſcribe valores variabilium conſequentes pro antecedentibus, hoc eſt, $T'T'$ pro TT, & $n+2$ pro n, & emerget

$$T'T' = A\overline{n+2} + B + \frac{C-2B}{n+4} + \frac{D-4C}{n+4.n+6} + \&c.$$

Tum ſumendo horum valorum differentiam, & ducendo eandem in $n+2$, prodibit

$$\overline{n+2} \times \overline{TT - T'T'} = -2A\overline{n+2} \quad * \quad + \frac{2C-4B}{n+4} + \frac{4D-16C}{n+4.n+6} + \&c.$$

Ceterum

Ceterum in valore ipſius T T prius aſſumpto, ſi ſcribatur $n+2$ pro n, habebitur

$$T'T' = \overline{n+2} \text{ in } A + \frac{B}{n+4} + \frac{C}{n+4.n+6} + \frac{D}{n+4.n+6.n+8} + \&c.$$

Atque dividendo per $n+2$,

$$\frac{T'T'}{n+2} = A + \frac{B}{n+4} + \frac{C}{n+4.n+6} + \frac{D}{n+4.n+6.n+8} + \&c.$$

Subſtitue jam in Æquatione reſolvenda, valores ipſorum $T'T'$, $\overline{n+2} \times \overline{TT - T'T'}$ & $\frac{T'T'}{n+2}$, ad eandem formam reductos, & reſultabit

$$2B - A + \frac{4C - 9B}{n+4} + \frac{6D - 25C}{n+4.n+6} + \frac{8E - 49D}{n+4.n+6.n+8} + \&c. = 0.$$

Denique ponendo Numeratores nihilo æquales, habebis $2B - A = 0$, $4C - 9B = 0$, $6D - 25C = 0$, $8E - 49D = 0$, &c. Et hæ ſunt relationes Coefficientium in prima Serie. Et ad eundem modum eruitur Series poſterior in Solutione priore.

Analyſis Solutionis poſterioris.

Solutio poſterior perficitur interpolando Seriem $1, \frac{1}{2}a, \frac{3}{4}b, \frac{5}{6}c, \frac{7}{8}d$, &c. cujus Termini ſunt reciproci eorum in priore: ea vero definitur Æquatione $T' = \frac{n+1}{n+2}T$, in qua valores ipſius n ſucceſſivi ſunt $0, 2, 4, 6, 8$, &c. ut ante. Et quadrando evadit $T'T' = \frac{nn+2n+1}{nn+4n+4}TT$, hoc eſt, $\overline{n+1}.\overline{n+3}\ \overline{TT - T'T'} - 2.\overline{n+1}\,TT - T'T' = 0$.

Fingamus nunc

$$TT = \frac{A}{n+1} + \frac{B}{n+1.n+3} + \frac{C}{n+1.n+3.n+5} + \frac{D}{n+1.n+3.n+5.n+7} + \&c.$$

Et ſubſtituendo $n+2$ pro n, proveniet

$$T'T' = \frac{A}{n+3} + \frac{B}{n+3.n+5} + \frac{C}{n+3.n+5.n+7} + \frac{D}{n+3.n+5.n+7.n+9} + \&c.$$

adeoque

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adeoque

$$\overline{n+1}\cdot\overline{n+3}\cdot TT - TT = 2A + \frac{4B}{n+5} + \frac{6C}{n+5\cdot n+7} + \frac{8D}{n+5\cdot n+7\cdot n+9} + \&c.$$

quæ ad debitam formam reducta ſit

$$2A + \frac{4B}{n+3} + \frac{6C - 8B}{n+3\cdot n+5} + \frac{8D - 49C}{n+3\cdot n+5\cdot n+7} + \&c.$$

Scribe jam hoſce valores in Æquatione reſolvenda, atque reſultabit

$$\frac{2B - A}{n+3} + \frac{4C - 9B}{n+2\cdot n+5} + \frac{6E - 25C}{n+3\cdot n+5\cdot n+7} + \frac{8E - 49D}{n+3\cdot n+5\cdot n+7\cdot n+9} + \&c. = 0.$$

Ubi Numeratores æquati nihilo, dabunt relationem Coefficientium Seriei prioris in Solutione poſteriore.

Quod autem Coefficiens A ſit in uno Caſu ſemicircumferentia Circuli, & ejuſdem reciproca in altero, ſic demonſtratur. Per Seriem priorem eſt $TT = An$ in $1 + \frac{1}{2\cdot n+2} + \&c.$ Et quo major eſt n, eo propius Æquatio $TT = An$, accedet ad veritatem, quoniam Termini poſteriores tandem evadent infinite minores prioribus. Igitur in Æquatione $TT = An$, vel $A = \frac{TT}{n}$, ſi pro n ſucceſſive ſcribantur ejus valores, 2, 4, 6, 8, 10, &c. & ſimul quadrata Terminorum correſpondentium pro TT; orientur Æquationes continue approximantes ad verum ipſius A valorem, ſcilicet

$$A = 2,$$
$$A = 2 \times \frac{8}{9},$$
$$A = 2 \times \frac{8}{9} \times \frac{24}{25},$$
$$A = 2 \times \frac{8}{9} \times \frac{24}{25} \times \frac{48}{49},$$
&c.

Quare valor ipſius A eſt factum ſub omnibus $2 \times \frac{8}{9} \times \frac{24}{25} \times \frac{48}{49} \times \frac{80}{81} \times \&c.$ uſque in infinitum; quod æquale eſt ſemicircumferentiæ Circuli per *Arithmeticam Infinitorum Walliſii.*

Propositio

Propositio XXIV.

Si in Ordinata Curvæ $x^{r+z-1} \times \overline{1-x}^{p-r-1}$ *ſcribantur ſucceſſive numeri integri* 0, 1, 2, 3, 4, *&c. pro z, dico eandem eſſe relationem inter Areas Ordinatarum provenientium, quæ eſt inter Terminos Seriei a,* $\frac{r}{p}a$, $\frac{r+1}{p+1}b$, $\frac{r+2}{p+2}c$, $\frac{r+3}{p+3}d$, *&c. modo abſciſſa x ſit æqualis unitati.*

Sint enim Areæ & Ordinatæ correſpondentes

Areæ	Ordinatæ
A	$x^{r-1} \times \overline{1-x}^{p-r-1}$
B	$x^{r} \times \overline{1-x}^{p-r-1}$
C	$x^{r+1} \times \overline{1-x}^{p-r-1}$
D	$x^{r+2} \times \overline{1-x}^{p-r-1}$
E	$x^{r+3} \times \overline{1-x}^{p-r-1}$
&c.	&c.

Tum comparando haſce Areas per Propoſitionem ſeptimam *Newtoni* de Quadraturâ Curvarum, invenies

$$B = \frac{rA - x^r \times \overline{1-x}^{p-r}}{p},$$

$$C = \frac{\overline{r+1}\,B - x^{r+1} \times \overline{1-x}^{p-r}}{p+1},$$

$$D = \frac{\overline{r+2}\,C - x^{r+2} \times \overline{1-x}^{p-r}}{p+2},$$

$$E = \frac{\overline{r+3}\,D - x^{r+3} \times \overline{1-x}^{p-r}}{p+3},$$

&c.

Sit jam x æqualis unitati, ut in Theoremate ponitur; eritque $1 - x = 0$; quo in caſu relatio Arearum fit $B = \frac{r}{p} A$, $C = \frac{r+1}{p+1} B$, $D = \frac{r+2}{p+2} C$, $E = \frac{r+3}{p+3} D$, &c. Adeoque eadem eſt relatio inter Areas harum Curvarum & Terminos Seriei propoſitæ, quando Abſciſſa x eſt unitas. Q. E. D.

Corollarium. Hinc in Serie a, $\frac{r}{p} a$, $\frac{r+1}{p+1} b$, $\frac{r+2}{p+2} c$, &c. ſi z denotet intervallum inter primum Terminum a, & alium quemvis T primarium vel intermedium; erit ut Area Curvæ cujus Ordinata eſt $x^{r-1} \times \overline{1-x}|^{p-r-1}$ ad Aream Curvæ cujus Ordinata eſt $x^{r+z-1} \times \overline{1-x}|^{p-r-1}$, ita Terminus primus a ad alium quemvis primarium vel intermedium, intervallo z diſtantem ab initio.

Exemplum I.

Detur Series intercalanda 1, $\frac{1}{2} a$, $\frac{3}{4} b$, $\frac{5}{6} c$, $\frac{7}{8} d$, &c. Quoniam differentia communis tum Numeratorum tum Denominatorum eſt 2; divide eoſdem per binarium, ut differentia illa fiat unitas ut in Theoremate; & evadet Series 1, $\frac{\frac{1}{2}}{1} a$, $\frac{\frac{3}{2}}{2} b$, &c. quæ collata cum ea in Propoſitione, dat $p = 1$, $r = \frac{1}{2}$; quibus ſubſtitutis, erit ut Area Ordinatæ $x^{-\frac{1}{2}} \times \overline{1-x}|^{-\frac{1}{2}}$ ad Aream Ordinatæ $x^{z-\frac{1}{2}} \times \overline{1-x}|^{-\frac{1}{2}}$, id eſt, ut Area hujus $\frac{1}{\sqrt{x-xx}}$ ad Aream hujus $\frac{x^z}{\sqrt{x-xx}}$ ita primus Terminus Seriei, ſive unitas ad alium quemvis primarium vel intermedium intervallo z diſtantem ab initio.

Ut ſi Terminus deſideratus conſiſtat in medio inter primum & ſecundum, erit $z = \frac{1}{2}$, quo in Caſu fit Ordinata poſterior $\frac{x^{\frac{1}{2}}}{\sqrt{x-xx}}$, ſive $\frac{1}{\sqrt{1-x}}$: adeoque ut Area Ordinatæ $\frac{1}{\sqrt{x-xx}}$ ad Aream Ordinatæ $\frac{1}{\sqrt{1-x}}$, id eſt, ut Circumferentia Circuli 3.1415926 - - - &c. ad 2, ita unitas ad Terminum in medio inter primum & ſecundum; qui hoc pacto prodit .63661977 - - - &c.

Si

Si ejuſdem Seriei quæratur Terminus centeſimus primus, erit $z = 100$; adeoque ut 3.1415 - - - &c. ad Aream Ordinatæ $\frac{x^{100}}{\sqrt{x-xx}}$, ita unitas ad Terminum Propoſitum. Et ſimiliter ponendo $z = 100\frac{1}{2}$, Terminus in medio inter centeſimum primum & centeſimum ſecundum determinabitur per Circumferentiam Circuli & Aream Ordinatæ $\frac{x^{100}}{\sqrt{1-x}}$: ubique tamen ſumendo partes Arearum quæ jacent ſupra Abſciſſam unitati æqualem.

Poſſunt etiam Terminorum reciproci interpolari, idque nonnunquam magis commode quam ipſi Termini. Reciproci Terminorum in noviſſima Serie ſunt 1, $\frac{2}{1}a$, $\frac{4}{3}b$, &c. adeoque eſt $r = 1$, $p = \frac{1}{2}$; & inde ut Area Ordinatæ $x^0 \times \overline{1-x}^{-\frac{1}{2}}$ ad Aream Ordinatæ $x^z \times \overline{1-x}^{-\frac{3}{2}}$ ita primus Terminus ad illum diſtantem intervallo z ab initio: id eſt, ut 2 ad Aream Ordinatæ $\frac{x^z}{\overline{1-x}\sqrt{1-x}}$; nam prior Curvæ eſt quadrabilis.

Exemplum II.

Interpolanda ſit Series 1, $\frac{1}{2}a$, $\frac{4}{5}b$, $\frac{7}{8}c$, $\frac{10}{11}d$, &c. Divide Numeratores & Denominatores per eorum incrementum 3; & invenies $p = \frac{2}{3}$, $r = \frac{1}{3}$; hinc fit Ordinata prior $x^{-\frac{2}{3}} \times \overline{1-x}^{-\frac{2}{3}}$, ſive $\frac{1}{\sqrt[3]{x^2 - 2x^3 + x^4}}$; & poſterior $\frac{x^z}{\sqrt[3]{x^2 - 2x^3 + x^4}}$. Dein ut Area illius ad Aream hujus ita primus Terminus Seriei ad alium quemcunque cujus diſtantia ab initio eſt z.

Propositio XXV.

Si in Ordinata Curvæ $x^{p-z} \times \overline{1-x}^{r+z-1}$ *ſcribantur ſucceſſive numeri integri* 0, 1, 2, 3, 4, *&c. pro z, eadem erit relatio inter Areas Ordinatarum provenientium* *quæ*

quæ eſt inter Terminos Seriei a, $\frac{r}{p}a$, $\frac{r+1}{p-1}b$, $\frac{r+2}{p-2}c$, $\frac{r+3}{p-3}d$, *&c. Ubi Numeratores continue increſcunt, Denominatores vero decreſcunt. Et hic quoque pono Abſciſſam* x *unitati æqualem.*

Hæc Propoſitio demonſtratur ad modum ſuperioris.

Corollarium. Hinc in Serie a, $\frac{r}{p}a$, $\frac{r+1}{p-1}b$, $\frac{r+2}{p-2}c$, $\frac{r+3}{p-3}d$, &c. ut Terminus primus a ad alium quemvis intervallo z diſtantem a principio, ita Area Curvæ cujus Ordinata eſt $x^p \times \overline{1-x}^{r-1}$ ad Aream Curvæ cujus Ordinata eſt $x^{p-z} \times \overline{1-x}^{r+z-1}$.

Exemplum I.

Detur Series interpolanda 1, $\frac{n}{1}a$, $\frac{n-1}{2}b$, $\frac{n-2}{3}c$, $\frac{n-3}{4}d$, &c. cujus Termini ſunt Unciæ binomii in dignitate cujus index eſt n. Quoniam hæc Series non cadit directe ſub hàc Propoſitione, interpolo Terminorum reciprocos 1, $\frac{1}{n}a$, $\frac{2}{n-1}b$, $\frac{3}{n-2}c$, &c. quo pacto erit $r=1$, $p=n$; adeoque ut Area Ordinatæ $x^n \times \overline{1-x}^0$ ad Aream Ordinatæ $x^{n-z} \times \overline{1-x}^z$, ita unitas ad Terminum intervallo z diſtantem a principio, in Serie poſteriore: vel ut Area Ordinatæ $x^{n-z} \times \overline{1-x}^z$ ad $\frac{1}{n+1}$, ita unitas ad Terminum in priore Serie, intervallo z remotum ab initio.

Ut ſi deſideretur Uncia Termini quinti in dignitate nona, ërit $n=9$, $z=4$; quibus ſcriptis, erit Area Ordinatæ $x^5 \times \overline{1-x}^4$ ad $\frac{1}{10}$ ut unitas ad Unciam quæſitam. Ordinata autem evoluta eſt $x^5 - 4x^6 + 6x^7 - 4x^8 + x^9$, ejuſque Area $\frac{1}{6} - \frac{4}{7} + \frac{6}{8} - \frac{4}{9} + \frac{1}{10}$, ſive $\frac{1}{1260}$; Tum ut $\frac{1}{1260}$ ad $\frac{1}{10}$ ita unitas ad 126, quæ eſt Uncia propoſita.

Exemplum

Exemplum II.

Si in poteſtate ſimplici Binomii, quæratur Terminus qui conſiſtit in medio inter duas Uncias 1 & 1; erit index binomii $n=1$, $z=\frac{1}{2}$; & exinde ut Area Ordinatæ $x^{\frac{1}{2}}\times\overline{1-x}^{\frac{1}{2}}$ ad $\frac{1}{2}$, hoc eſt, ut Area Circuli ad quadratum Circumſcriptum ita unitas ad Terminum inter Uncias 1 & 1.

Scholion.

Quando Curvæ quadrandæ ſunt plurimarum dimenſionum, inveniendæ ſunt aliquot earum Ordinatæ per Tabulam Logarithmorum; ex quibus dabuntur Areæ per Parabolam *Newtoni*. Quod ſi relatio Terminorum in Serie interpolanda ſit inter plures Terminos, abſolvetur interpolatio per Comparationem aliarum Curvarum. Sed hanc materiam miſſam faciens adjungam quædam de aliis methodis interpolandi.

Propositio XXVI.

Si Series interpolanda ſit $1,\ \frac{r}{p}a,\ \frac{r+1}{p+1}b,\ \frac{r+2}{p+2}c,\ \frac{r+3}{p+3}d,$ *&c. & ponatur* $n=r-p$, *atque*

$$B=\frac{n}{1}\cdot\frac{n-1}{2}\ \textit{in}\ A,$$

$$C=\frac{n-1}{2}\cdot\frac{n-2}{2}\ \textit{in}\ B+\frac{n}{3}A,$$

$$D=\frac{n-2}{3}\cdot\frac{n-3}{2}\ \textit{in}\ C+\frac{n-1}{3}B+\frac{n}{3}\cdot\frac{n-1}{4}A,$$

$$E=\frac{n-3}{4}\cdot\frac{n-4}{2}\ \textit{in}\ D+\frac{n-2}{3}C+\frac{n-1}{3}\cdot\frac{n-2}{4}B+\frac{n}{3}\cdot\frac{n-1}{4}\cdot\frac{n-2}{5}A,$$

$$F=\frac{n-4}{5}\cdot\frac{n-5}{2}\ \textit{in}\ E+\frac{n-3}{3}D+\frac{n-2}{3}\cdot\frac{n-3}{4}C+\frac{n-1}{3}\cdot\frac{n-2}{4}\cdot\frac{n-3}{5}B$$
$$+\frac{n}{3}\cdot\frac{n-1}{4}\cdot\frac{n-2}{5}\cdot\frac{n-3}{6}A,$$

&c.

Tum

Tum erit z^n *in* $A+\frac{B}{z}+\frac{C}{z^2}+\frac{D}{z^3}+\frac{E}{z^4}+$ *&c. Terminus Seriei interpolandæ primarius vel intermedius, cujus diſtantia ab initio eſt* $z-p$.

Notandum eſt Coefficientem A, per Propoſitionem decimam octavam æqualem eſſe Termino in Serie Numeratorum 1, ra, $\overline{r+1}.b$, $\overline{r+2}.c$, &c. diſtantem intervallo $p-r$ ab initio; eamque determinari per Exemplum ſecundum Propoſitionis vigeſimæ primæ.

Demonstratio.

Series propoſita definitur Æquatione differentiali $T'=\frac{z+n}{z}T$, ubi eſt $n=r-p$ ut in Theoremate & ſucceſſivi valores indeterminatæ z ſunt p, $p+1$, $p+2$, &c. Finge jam

$$T=z^n \text{ in } A+\frac{B}{z}+\frac{C}{z^2}+\frac{D}{z^3}+\frac{E}{z^4}+\&c.$$

dein pro T & z ſcribe earum valores ſuccedentes T' & $z+1$, reſpective, & emerget

$$T'=\overline{z+1}^n \text{ in } A+\frac{B}{z+1}+\frac{C}{\overline{z+1}^2}+\frac{D}{\overline{z+1}^3}+\frac{E}{\overline{z+1}^4}+\&c.$$

Vel evolvendo dignitates

$$T'=z^n \text{ in } A+ \frac{nA+B}{z}+\frac{\overline{nA+2B}.\overline{n-1}+2C}{z^2}+\frac{\overline{nA+3B}.\overline{n-1}.\overline{n-2}+6.\overline{n-2}.C+6D}{z^3}+\&c.$$

Æquatio reſolvenda $T'=\frac{z+n}{z}T$, hunc in modum ſcribatur $T'z-Tz-Tn=0$; & in eadem ſubſtituantur valores ipſorum T & T', atque reſultabit

$$z^n \text{ in } \frac{n.\overline{n-1}A-2B}{2z}+\frac{\overline{nA+3B}.\overline{n-1}.\overline{n-2}-12C}{6z^2}+\&c.=0.$$

Ponantur jam Numeratores æquales nihil, & habebuntur $B=\frac{n}{1}.\frac{n-1}{2}$ in A, $C=\frac{n-1}{2}.\frac{n-2}{2}$ in $B+\frac{n}{3}A$, &c. Atque producendo computum, reliquæ Coefficientes prodibunt in Theoremate. Q. E. D.

Exemplum

Exemplum I.

Proponatur Series 1, $\frac{1}{2}a$, $\frac{3}{4}b$, $\frac{5}{6}c$, $\frac{7}{8}d$, &c. quæ definitur Æquatione $T' = \frac{z-\frac{1}{2}}{z}T$, in qua valores ſucceſſivi ipſius z ſunt 1, 2, 3, 4, &c. hæc comparata cum Æquatione $T' = \frac{z-n}{z}T$, dat $n = -\frac{1}{2}$; & exinde

$$B = \frac{3}{8} \text{ in } A \qquad = A \times \frac{3}{8},$$

$$C = \frac{15}{16} \text{ in } B - \frac{1}{6}A \qquad = A \times \frac{25}{128},$$

$$D = \frac{35}{24} \text{ in } C - \frac{3}{6}B + \frac{1}{16}A \qquad = A \times \frac{105}{1024},$$

$$E = \frac{63}{32} \text{ in } D - \frac{5}{6}C + \frac{5}{16}B - \frac{1}{32}A \qquad = A \times \frac{1659}{32768},$$

$$F = \frac{99}{40} \text{ in } E - \frac{7}{6}D + \frac{35}{48}C - \frac{7}{32}B + \frac{7}{384}A \qquad = A \times \frac{6237}{262144},$$

&c.

Adeoque

$$T = \frac{A}{\sqrt{z}} \text{ in } 1 + \frac{3}{8z} + \frac{25}{128z^2} + \frac{105}{1024z^3} + \frac{1659}{32768z^4} + \frac{6237}{262144z^5} + \&c.$$

Quantitas autem A in hujuſmodi Exemplis ſic determinari poſſit. Per relationem Terminorum interpolandorum quære primarium quemvis ſatis longe diſtantem ab initio, verbi gratiâ decimum ſextum, qui hic prodit .144464448 - - - &c. Hunc ſcribe pro T & interea pro z valorem ſuum correſpondentem, nempe 16; atque habebis

$$.144464448 = \frac{A}{4} \text{ in } 1 + \frac{3}{8.16} + \frac{25}{128.16.16} + \&c.$$

vel colligendo Terminos in unam Summam, $.144464448 = \frac{A}{4}$ in 1.02422627, unde prodit A = .564189583548: quo dato dabitur T in alio quovis caſu per Terminos pauciſſimos ſui valoris.

Exemplum

Exemplum II.

Interpolanda ſit Series $1, \frac{2}{1}a, \frac{5}{4}B, \frac{8}{7}c, \frac{11}{10}d,$ &c. quæ definitur Æquatione $T' = \frac{z+\frac{1}{3}}{z}T$, exiſtentibus $\frac{1}{3}, \frac{4}{3}, \frac{7}{3}, \frac{10}{3}$, &c. valoribus Abſciſſæ z ſucceſſivis: hæc comparata cum Æquatione in Theoremate dat $n = \frac{1}{3}$; quo ſubſtituto obtinebitur

$$T = A\sqrt[3]{z} \text{ in } 1 - \frac{1}{9z} \; * + \frac{5}{2187z^3} + \frac{253}{19683z^4} + \frac{5610}{177147z^5} + \&c.$$

Jam ad eruendum Coefficientem A, quæro Terminum decimum quartum Seriei interpolandæ, qui prodit 4.652136: hunc deinde valorem ſcribo pro T, & pro z valorem ſuum decimum quartum $\frac{40}{3}$, atque habeo

$$4.652136 = A\sqrt[3]{\frac{40}{3}} \text{ in } 1 - \frac{1}{120} + \frac{1}{345600} + \&c.$$

Vel extrahendo radicem cubicam numeri $\frac{40}{3}$, & colligendo Terminos in unam ſummam, obtineo $4.652136 = A \times 2.351505$, adeoque $A = 1.978365$. Datâ nunc A, Terminus quilibet alius invenietur ſumma facilitate. Quæratur is qui conſiſtit tertiâ parte intervalli communis ante milleſimum primum: pro z ſcribe ſuum valorem correſpondentem 1000, & valor ipſius T prodibit $10\,A$ in $1 - \frac{1}{9000}$, ſive $T = 19.78343$. Nam ubi Terminus deſideratus longe diſtat a principio & computus non eſt producendus ad plurimas figuras perpauci Termini in valore ipſius T abunde ſufficiunt.

Scholion.

Eodem modo quo in hâc Propoſitione Radix extrahitur ex Æquatione $T' = \frac{z+n}{z}T$, extrahitur etiam ex aliâ quâvis quæ ſub hâc formâ continetur

$$T \times \overline{z^l + az^{l-1} + bz^{l-2} + \&c.} = T' \times \overline{z^l + cz^{l-1} + dz^{l-2} + \&c.}$$

Nam

Nam eſt index $n = a - c$ per Propoſitionem ſextam; & inde forma Seriei pro T aſſumenda erit

$$T = Az^{n} + Bz^{n-1} + Cz^{n-2} + \&c.$$

Quippe in hujuſmodi Seriebus quæ ſunt Radices Æquationum Differentialium, indices ipſius z habent Unitatem pro decremento, prætérquam in quibuſdam caſibus valde particularibus. Habito igitur indice ipſius z in primo Termino habetur forma Seriei aſſumendæ pro Radice T: dein ſcribendo $z+1$ pro z, & T′ pro T, prodibit

$$T' = A\overline{z+1}^{n} + B\overline{z+1}^{n-1} + C\overline{z+1}^{n-2} + \&c.$$

Tum reducendo hunc valorem ad formam ipſius T, ut ſupra oſtenſum eſt, & ducendo utrumque valorem in quantitates quas præcipit Æquatio reſolvenda; dabuntur Coefficientes aſſumptæ ex collatione membrorum homologorum in Æquatione reſultante

Propositio XXVII.

Si Æquatio ad Seriem ſit $T' = \frac{zz}{zz+r}T$, *erit Radix*

$$T = A + \frac{r}{z}A + \frac{r+1}{2.\overline{z+2}}B + \frac{r+4}{3.\overline{z+3}}C + \frac{r+9}{4.\overline{z+4}}D + \frac{r+16}{5.\overline{z+5}}E + \&c.$$

Æquatio reſolvenda $T' = \frac{zz}{zz+r}T$ hunc in modum ſcribatur $zzT - zzT' - rT' = 0$, & aſſumatur

$$T = A + \frac{B}{z} + \frac{C}{z.\overline{z+1}} + \frac{D}{z.\overline{z+1}.\overline{z+2}} + \frac{E}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

Eritque

$$T' = A + \frac{B}{z+1} + \frac{C}{\overline{z+1}.\overline{z+2}} + \frac{D}{\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{E}{\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$$

Unde

$$T - T' = \frac{B}{z.\overline{z+1}} + \frac{2C}{z.\overline{z+1}.\overline{z+2}} + \frac{3D}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{4E}{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$$

Et ducendo in zz

$$zzT - zzT' = \frac{Bz}{z+1} + \frac{2Cz}{\overline{z+1}.\overline{z+2}} + \frac{3Dz}{\overline{z+1}.\overline{z+2}.\overline{z+3}} + \frac{4Fz}{\overline{z+1}.\overline{z+2}.\overline{z+3}.\overline{z+4}} + \&c.$$

Atque per reductionem

$$zzT - zzT' = B + \frac{2C - B}{z+1} + \frac{3D - 4C}{\overline{z+1}.\overline{z+2}} + \frac{4E - 9D}{\overline{z+1}.\overline{z+2}.\overline{z+3}} + \&c.$$

 Hunc

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Hunc valorem ſcribe pro $zzT - zzT'$, & pro T' ſuum valorem hactenus repertum, atque reſultabit

$$B - rA + \frac{2C - \overline{r+1}B}{z+1} + \frac{3D - \overline{r+4}C}{z+1.z+2} + \frac{4E - \overline{r+9}D}{z+1.z+2.z+3} + \&c. = 0.$$

Et per collationem membrorum homologorum, $B = \frac{r}{1}A$, $C = \frac{r+1}{2}B$, $D = \frac{r+4}{3}C$, $E = \frac{r+9}{4}D$, &c. Hi ſunt valores Coefficientium: quod ſi A, B, C, &c. denotent totos Terminos, prodibit valor ipſius T jam aſſignatus. Q. E. D.

Exemplum.

Invenit *Walliſius*, ultimum Terminum hujus Seriei 1, $\frac{8}{9}A$, $\frac{24}{25}B$, $\frac{48}{49}C$, $\frac{80}{81}D$, &c. eſſe Aream Circuli cujus Diameter eſt unitas; ubi Denominatores ſunt quadrata numerorum imparium, & unitate majores Numeratoribus. Videamus autem hic quiſnam ſit ultimus Terminus hujus 1, $\frac{4}{3}A$, $\frac{16}{15}B$, $\frac{36}{35}C$, $\frac{64}{63}D$, &c. ubi Numeratores ſunt quadrata numerorum parium & unitate majores Denominatoribus. Æquatio ad hanc Seriem erit $T' = \frac{zz}{zz - \frac{1}{4}}T$, exiſtentibus 1, 2, 3, 4, &c. valoribus Abſciſſæ z ſucceſſivis: ergo comparando hanc Æquationem cum illâ in Propoſitione, erit $r = -\frac{1}{4}$, quo ſubſtituto prodit

$$T = A - \frac{A}{4z} + \frac{3B}{8.\overline{z+1}} + \frac{15C}{12.\overline{z+2}} + \frac{35D}{16.\overline{z+3}} + \frac{63E}{20.\overline{z+4}} + \&c.$$

Et ad determinandum Coefficientem A, quære Terminum Seriei decimum, nempe 1.5300.1727.35, quem ſubſtitue pro T, & interea pro z valorem ſuum correſpondentem 10; atque obtinebis

$$1.5300.1727.35 = A \text{ in } 1 - \frac{1}{4.10} - \frac{1.3}{4.8.10.11} - \frac{1.3.15}{4.8.12.10.11.12} - \&c.$$

Hoc eſt, colligendo Terminos in unum, 1.5300.1727.35 = A in .9740.392454, & exinde A = 1.57079633, utique ſemicircumferentiæ Circuli: quo dato, facillime dabitur Terminus quilibet Seriei interpolandæ primarius vel intermedius. Conſtat autem ultimum ejus Terminum,

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minum, ſive factum ſub omnibus $\frac{4}{3} \times \frac{16}{15} \times \frac{36}{35} \times \frac{64}{63} \times \frac{100}{99} \times$ &c. æquale eſſe primo Coefficienti A, adeoque ſemiperiferiæ Circuli.

PROPOSITIO XXVIII.

Invenire ſummam quotcunque Logarithmorum, quorum numeri ſunt in progreſſione Arithmetica.

Deſignent $x+n$, $x+3n$, $x+5n$, $x+7n$, --- $z-n$, quotcunque numeros in progreſſione Arithmeticâ, quorum primus & $x+n$, ultimus $z-n$, & communis differentia $2n$. Inſuper denotent l,z & l,x Logarithmos Tabulares numerorum z & x; ſitque $a = .43429.44819.03252$, ſcilicet reciproco Logarithmi naturalis Denarii. Atque ſumma Logarithmorum propoſitorum æqualis erit differentiæ inter Series duas ſequentes

$$\frac{zl,z}{2n} - \frac{az}{2n} - \frac{an}{12z} + \frac{7an^3}{360z^3} - \frac{31an^5}{1260z^5} + \frac{127an^7}{1680z^7} - \frac{511an^9}{1188z^9} + \&c.$$
$$\frac{xl,x}{2n} - \frac{ax}{2n} - \frac{an}{12x} + \frac{7an^3}{360x^3} - \frac{31an^5}{1260x^5} + \frac{127an^7}{1680x^7} - \frac{511an^9}{1188x^9} + \&c.$$

Hæ autem Series ſic continuantur in infinitum; pone

$$-\frac{1}{3.4} = A,$$
$$-\frac{1}{5.8} = A + 3B,$$
$$-\frac{1}{7.12} = A + 10B + 5C,$$
$$-\frac{1}{9.16} = A + 21B + 35C + 7D,$$
$$-\frac{1}{11.20} = A + 36B + 126C + 84D + 9F,$$
&c.

Ubi numeri qui multiplicantur in A, B, C, D, &c. in diverſis valoribus ſunt Unciæ alternæ in dignitatibus imparibus binomii. Hiſce præmiſſis, erit Coefficiens Termini tertii $-\frac{1}{12} = A$, is quarti $+\frac{7}{360} = B$, quinti $-\frac{31}{1260} = C$, & ſic deinceps.

DEMON-

Demonstratio.

Minuatur variabilis z decremento ſuo $2n$; vel quod idem eſt, ſubſtituatur $z - 2n$ pro z in Serie

$$\frac{zl,z}{2n} - \frac{az}{2n} - \frac{an}{12z} + \frac{7an^3}{360z^3} - \frac{31an^5}{1260z^5} + \&c.$$

& proveniet valor ejuſdem ſucceſſivus

$$\frac{\overline{z-2n}\,l,\overline{z-2n}}{2n} - \frac{a}{2n} \times \overline{z-2n} - \frac{an}{12.\overline{z-2n}} + \frac{7an^3}{360.\overline{z-2n}^3} - \frac{31an^5}{1260.\overline{z-2n}^5} + \&c.$$

Hunc ſubducito de valore priore, Terminis prius ad eandem formam per Diviſionem reductis, & relinquetur $lz - \frac{an}{z} - \frac{an^2}{2z^2} - \frac{an^3}{3z^3} - \frac{an^4}{4z^4} -$ &c. id eſt, Logarithmus numeri $z - n$. Adeoque univerſaliter decrementum duorum valorum ſucceſſivorum Seriei, æquatur Logarithmo ipſius $z - n$; qui exprimit in genere quemvis Logarithmorum qui erant ſummandi. Igitur Series erit ſumma Logarithmorum propoſitorum, ſi ab eâdem ſubducatur altera Series. Nam ſummæ perinde ac Areæ nonnunquam corrigendæ ſunt, ut evadant veræ.

Exemplum I.

Proponatur invenire ſummam Logarithmorum decem numerorum 101, 103, 105, 107, 109, 111, 113, 115, 117, 119; hi collati cum hiſce $x+n$, $x+3n$, $x+5n$, - - - - $z-n$, dant differentiam communem $2n=2$, & $n=1$; atque primum $x+1=101$, ſecundum vero $z-1=119$: unde $x=100$, $z=120$. Hiſce autem ſubſtitutis, & .43429.44819.03252 pro a; atque Logarithmis ipſorum 100 & 120 reſpective pro l, x & l, z; valores duarum Serierum invenientur 78.28491.40012.1 & 98.69290.42601.6, quorum differentia dat 20.40799.02589.5 pro Summa Logarithmorum deſiderata.

Exemplum II.

Quæratur jam Summa Logarithmorum numerorum 11, 12, 13, - - - 1000, quorum primus eſt 11, & ultimus 1000, & differentia communis unitas. Eſt igitur $n=\frac{1}{2}$, $x+\frac{1}{2}=10$, $z-\frac{1}{2}=1000$; unde $x=\frac{21}{2}$, $z=\frac{2001}{2}$; quibus ſcriptis, & Logarithmis ipſorum $\frac{21}{2}$ & $\frac{2001}{2}$ pro l, x & l, z, prodeunt pro valoribus Serierum 2567.20555.42879 &
6.16067.309 87,

6.16067.30987, quorum differentia relinquit 2561.04488.11892 pro Summâ Logarithmorum quæſitâ.

Ceterum ſi velis ſummam quotcunque Logarithmorum numerorum naturalium 1, 2, 3, 4, 5, &c. pone $z-n$ eſſe ultimum numerorum, exiſtente $r=\frac{1}{2}$; & tres vel quatuor Termini hujus Seriei $zl, z-az-\frac{a}{24z}+\frac{7a}{2880z^3}-$ &c. additi Logarithmo circumferentiæ Circuli cujus Radius eſt Unitas, id eſt, huic 0.39908.99341.79 dabunt ſummam quæſitam, idque eo minore labore quo plures Logarithmi ſunt ſummandi. Sic ſi ponas $z-\frac{1}{2}=1000$, vel $z=\frac{2001}{2}$ valor Seriei erit 2567.20555.42879 ut antea; qui adjectus Logarithmo conſtanti efficit 2567.60464.42221 pro ſummâ Logarithmorum, primorum mille numerorum hujus Seriei 1, 2, 3, 4, 5, &c

EXEMPLUM III.

Oporteat invenire Unciam quingenteſimam in dignitate Binomii milleſimâ. Conſtat per Theorema *Newtoni* pro evolvendo Binomio Unciam illam æqualem eſſe producto ſub quadringenti & nonaginta novem factoribus $\frac{1000}{1}, \frac{999}{2}, \frac{998}{3}, \frac{997}{4}, \frac{996}{5}, \ldots \frac{502}{499}$; quorum primus eſt $\frac{1000}{1}$, & ultimus $\frac{502}{499}$; exiſtentibus tum Numeratoribus tum Denominatoribus in progreſſione Arithmetica. Ad inveniendam ſummam Logarithmorum Numeratorum 1000, 999, 998, 997, ... 502; pone differentiam communem $1=2n$, maximum eorum $1000=z-\frac{1}{2}$, minimum $502=x+\frac{1}{2}$; eritque $n=\frac{1}{2}$, $z=1000\frac{1}{2}$, $x=501$: quibus ſubſtitutis, prodibit 2567.20555.42879 pro valore prioris Seriei, & 1136.38715.63268 pro valore poſterioris: horum autem differentia 1430.81839.79611 æqualis eſt ſummæ Logarithmorum Numeratorum. Dein ut obtineatur ſumma Logarithmorum Denominatorum 1, 2, 3, 4, ... 499; pone $n=\frac{1}{2}$, $z-\frac{1}{2}=499$, ſive $z=499\frac{1}{2}$; & hiſce ſcriptis in Serie priore, prodibit ejus valor 1130.98834.85966, cui adjiciatur Logarithmus .39908.99342 juxta regulam in Exemplo priore traditam: & habebis 1131.38743.85308 pro ſumma Logarithmorum Denominatorum. Quam denique ſubducito de ſumma Logarithmorum Numeratorum, atque manebit 299.43095.94303 Logarithmus ſcilicet Unciæ deſideratæ.

N n SCHOLIUM.

Scholion.

Hujuſmodi Series $1,\ \frac{r}{p}A,\ \frac{r+1}{p+1}B,\ \frac{r+2}{p+2}C$, &c. interpolatur per Propoſitionem vigeſimam ſextam, ubi differentia inter r & p eſt exigua: & per hanc Propoſitionem generaliter nullâ ratione habitâ iſtius differentiæ. Atque eodem plane modo invenire licet ſummam Logarithmorum Numerorum, qui ſunt longe magis compoſiti quam æquidifferentes; eoque pacto aſſignare Terminos Serierum quarum intercalatio pro difficillima haberi ſolet. Per hoc quoque Problema inveniuntur Areæ Curvarum quarum Ordinatæ ſunt hujuſmodi $\overline{1-x}^{1000}$, ubi index Binomii eſt permagnus; ſed in eo caſu ſolo quando pars Areæ quæſita jacet ſupra partem Abſciſſæ æqualem Unitati.

Et ſane omnia fere Problemata de Interpolationibus huic Analyſi ſubjiciuntur, imo etiamſi tres vel plures Termini Seriei intercalandæ ingrediantur Æquationem Differentialem: harum enim reſolutionem in poteſtate habeo. Atque hic notare libet Series quæ prodeunt per Parabolam *Newtoni*, prodire etiam per noſtram methodum. Proponatur enim intercalatio Seriei

$$a,\ \frac{r+n}{r}a,\ \frac{r+n+1}{r+1}b,\ \frac{r+n+2}{r+2}c,\ \frac{r+n+3}{r+3}d,\ \&c.$$

quæ definitur Æquatione $T'=\frac{z+n+r}{z+r}T$, in qua valores Abſciſſæ z ſucceſſivi ſunt o, 1, 2, 3, &c. Ea vero ſcribatur in modum ſequentem $\overline{z+r}\times\overline{T'-T}-nT=0$; & fingatur

$$T=A+Bz+Cz.\overline{z-1}+Dz.\overline{z-1}.\overline{z-2}+Ez.\overline{z-1}.\overline{z-2}.\overline{z-3}+\&c.$$

Atque ſubſtituendo T' pro T, & $z+1$ pro z, obtinebitur

$$T'=A+B\overline{z+1}.+C\overline{z+1}.z+D\overline{z+1}.z.\overline{z-1}+E\overline{z+1}.z.\overline{z-1}.\overline{z-2}+\&c.$$

Unde $T'-T=B+2Cz+3Dz.\overline{z-1}+4Ez.\overline{z-1}.\overline{z-2}+\&c.$

Hiſce autem valoribus ſubſtitutis in Æquatione reſolvenda, & membris reductis ad eandem formam, reſultabit

$$\left.\begin{array}{l}+rB\\-nA\end{array}\right\}\left.\begin{array}{l}+2.\overline{r+1}C\\-\overline{n-1}B\end{array}\right\}z\left.\begin{array}{l}+3.\overline{r+2}D\\-\overline{n-2}C\end{array}\right\}z.\overline{z-1}\left.\begin{array}{l}+4.\overline{r+3}E\\-\overline{n-3}D\end{array}\right\}z.\overline{z-1}.\overline{z-2}+\&c.=0.$$

Denique ponendo membra homologa æqualia nihil, habebitur $B=\frac{n}{r}A$, $C=\frac{1}{2}\times\frac{n-1}{r+1}B$, $D=\frac{1}{3}\times\frac{n-2}{r+2}C$, $E=\frac{1}{4}\times\frac{r-3}{r+3}D$, &c.

Adeoque

T=

$$T = A + A \times \frac{n}{r} \times \frac{z}{1} + A \times \frac{n}{r} \times \frac{z}{1} \times \frac{n-1}{r+1} \times \frac{z-1}{r+2} + \&c.$$

Hoc eſt

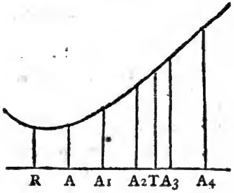
$$T = A + \frac{n}{r} A \frac{z}{1} + \frac{n-1}{r+1} B \frac{z-1}{2} + \frac{n-2}{r+2} C \frac{z-2}{3} + \frac{n-3}{r+3} D \frac{z-3}{4} \&c.$$

Ubi A, B, C, D, &c. non amplius denotant Coefficientes, ſed Terminos totos. Et prima Coefficiens A quæ ex Æquatione non determinatur, æqualis eſt Termino Seriei interpolandæ qui tranſit per initium Abſciſſæ *z*. Et hic eſt ipſiſſimus valor Termini T qui prodiiſſet per Propoſitionem decimam nonam. Atque etiam Series quas exhibet Propoſitio vigeſima invenientur per reſolutionem Æquationum Differentialium, aſſumendo debitam formam radicis.

Propositio XXIX.

Detur Series Ordinatarum intervallis quibuſcunque ab invicem diſtantium, pergens vero ex una tantum parte in infinitum; & oporteat invenire Lineam Parabolicam quæ tranſit per extremitates omnium.

Sint A, A1, A2, A3, A4, &c. Ordinatæ inſiſtentes Abſciſſæ in angulis rectis; ſitque R punctum quodlibet in Abſciſſâ; & ponatur $a =$ RA, $b =$ RA1, C=RA2, $d =$ RA3, $e =$ RA4, &c. ſcilicet ſint *a*, *b*, *c*, *d*, *e*, &c. intervalla inter Ordinatas & punctum R reſpective. Atque deſignet T Ordinatam quamvis in genere cujus diſtantia a puncto R ſit *z*. Dein ponatur



B=

$$B=\frac{A1-A}{b-a},\quad C=\frac{B1-B}{c-a},\quad D=\frac{C1-C}{d-a},\ E=\frac{D1-D}{e-a},\ \&c.$$

$$B1=\frac{A2-A1}{c-b},\ C1=\frac{B2-B1}{d-b},\ D1=\frac{C2-C1}{e-a},\quad \&c.$$

$$B2=\frac{A3-A2}{d-c},\ C2=\frac{B3-B2}{e-c},\qquad \&c.$$

$$B3=\frac{A4-A3}{e-d},\qquad \&c.$$

&c.

Atque erit Ordinata $T=A+$

$$B\times\overline{z-a}+$$
$$C\times\overline{z-a}\times\overline{z-b}+$$
$$D\times\overline{z-a}\times\overline{z-b}\times\overline{z-c}+$$
$$E\times\overline{z-a}\times\overline{z-b}\times\overline{z-c}\times\overline{z-d}+$$
$$F\times\overline{z-a}\times\overline{z-b}\times\overline{z-c}\times\overline{z-d}\times\overline{z-e}+$$

&c.

Notandum eſt principium Abſciſſæ, nempe punctum R ſumi ad arbitrium, vel inter Ordinatas vel extra omnes ut in Schemate; dummodo ſignorum $+$ & $-$ debita habeatur ratio. Propoſitio autem demonſtratur ſubſtituendo Ordinatas A, A1, A2, &c. ſucceſſive pro T, & interea pro *z* longitudines ejus ordine ſuccedentes *a*, *b*, *c*, &c. Nam ſumendo differentias Æquationum provenientium & dividendo eaſdem per intervalla Ordinatarum, obvenient valores Coefficientium ſupra aſſignati.

Exemplum I.

Sint Ordinatarum a principio Abſciſſæ, intervalla $a=2$, $b=3$, $c=5$, $d=6$; ipſa vero Ordinatæ ſint

$A=2$, $B=1$, $C=0$, $D=\frac{1}{2}$,
$A1=3$, $B1=1$, $C1=2$,
$A2=5$, $B2=7$,
$A3=12$.

Et ineundo computum juxta præſcripta Theorematis, invenietur $B=1$, $C=0$, $D=\frac{1}{2}$; quibus ſubſtitutis & 2 pro A, invenietur $T=2+\overline{z-2}+\frac{1}{2}\times\overline{z-2}.\overline{z-3}.\overline{z-5}$, quæ in ordinem reducta fit $T=$

$z^3-10.$

$\frac{z^3-10z^2+33z-30}{2}$. Etenim ſi inibi ſcribantur 2, 3, 5, 6, pro z; prodibunt Ordinatæ propoſitæ 2, 3, 5, 12.

Exemplum II.

Oporteat determinare tempus Solſtitii ex datis aliquot Solis altitudinibus meridianis circa idem tempus. Deſignent Ordinatæ altitudines ſolis, earumque intervalla denotent tempora inter obſervationes; tum tranſeat Parabola per extremitates Ordinatarum, & Abſciſſa ejus quæ correſpondet minimæ Ordinatæ, ſive ea ſit una e datis vel quævis intermedia, determinabit momentum temporis quo Sol ingreditur Tropicum. Exempli gratia *B. Walterus* Anno 1500, *Norimbergæ* obſervavit diſtantias ſolis a Zenith ut ſequitur

44975	8vo	Junii
44934	9no	
44883	12mo	
44990	16to	

Sit jam diſtantia obſervata die octavo Ordinata prima, & in eâdem ſit principium Abſciſſæ; eritque $a=0$, $b=1$, $c=4$, $d=8$; & computus erit

$$A=44975,\quad B=-41,\quad C=+6,\quad D=+\frac{1}{32},$$
$$A1=44934,\quad B1=-17,\quad C1=+\frac{175}{28},$$
$$A2=44883,\quad B2=+\frac{107}{4},$$
$$A3=44990,$$

Atque ſubſtituendo hoſce valores pro A, B, C, D; invenietur $T= 44975-41z+6z.\overline{z-1}+\frac{1}{32}z.\overline{z-1}.\overline{z-4}$, hoc eſt, $T=44975-\frac{375}{8}z+\frac{187}{32}z^2+\frac{1}{32}z^3$. Jam quoniam Abſciſſa quæſita correſpondet Ordinatæ minimæ, ponatur Fluxio ipſius T æqualis nihil, & habebitur $3zz+374z=1500$, cujus Radix 3.889355 exprimit dies elapſos inter meridiem octavi diei Junii & momentum Solſtitii: quod itaque accidit 21 hor. $20\frac{2}{3}$ min. pr. poſt meridiem diei decimæ primæ, ſecundum haſce Obſervationes. Poteſt etiam tempus Solſtitii determinari per plures Obſervationes & Parabolam plurium dimenſionum; vel per tres

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Obſervationes adhibendo Parabolam Conicam, uti docuit *Halleius.* Sed oportet differentias inter altitudines obſervatas, eſſe ſenſibiliter majores erroribus qui committi queant inter obſervandum, alias nihil certi concludi poterit.

Scholion.

Newtonus utitur hac Propoſitione in determinando locum Cometæ qui cadit inter aliquot loca Obſervationibus nota. Nimirum ſi obſerventur quotvis longitudines deſignatæ per totidem Ordinatas, quarum intervalla proportionalia ſunt temporibus inter Obſervationes, & deſcribatur Parabola per Ordinatarum extremitates; Ordinatæ hujus figuræ intermediæ denotabunt longitudines Cometæ intermedias pro temporibus quæ ſunt proportionalia Abſciſſis. Et eadem methodo dabitur latitudo pro quovis tempore ex datis aliquot latitudinibus. Ex longitudine autem & latitudine datis datur locus Cometæ in Cælis. Et hunc in modum plurima obſervatu difficilia determinari poſſint ſatis accurate ex Obſervationibus aliquot anterioribus & poſterioribus.

Applicabilis eſt etiam hæc Propoſitio ad reſolutionem Æquationum purarum affectarumve. Nam in Æquatione reſolvenda ſcribendo pro radice numeros ab eadem haud multum diſcrepantes; provenient eorum intervalla, qui dein interpolati exhibebunt radicem. Sed poſt reſolutionem Æquationum *Hallei* fruſtra compendioſior ſperanda eſt.

In caſu quando intervalla Ordinatarum diminuuntur in infinitum hoc Problema dabit radicem Æquationis fluxionalis, etiamſi nec radix neque altera indeterminata fluat uniformiter; & hoc merâ ſubſtitutione fluxionum Radicis pro differentiis Ordinatarum, & pro earum intervallis fluxiones Abſciſſæ. Nam ſicuti caſus Ordinatarum æquidiſtantium reſpondet fluxionibus Abſciſſæ uniformiter creſcentibus; ita hæc Propoſitio fluxionibus quâcunque lege variantibus. Et reſolutio Æquationis fluxionalis in quâ utraque indeterminata fluit quâcunque lege, non eſt Corollarium ex hac Propoſitione, ſed Caſus ejuſdem omnium ſimpliciſſimus: quod obiter hic monere viſum eſt, ut pateat Methodum Differentialem generaliſſime complecti univerſam Serierum doctrinam, quod forte alii non perceperint.

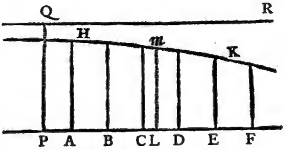
Propositio XXX.

Invenire Aſymptoton Hyperbolæ generis Logarithmici ex datis ejus Ordinatis aliquot æquidiſtantibus.

Sint Ordinatæ æquidiſtantes A, B, C, D, E, &c. inſiſtentes ſuper Abſciſſam PL in angulis rectis: ſitque QR Aſymptotos Curvæ parallela

rallela Abſciſſæ, & intervallo P Q ab eâdem diſtans. Apelletur vero Abſciſſa quævis A L z, & Ordinata correſpondens L m y. Hyperbola autem Logarithmica H m K definiatur Æquatione hujus formæ

$$y = a - br^{z} - cr^{2z} - dr^{3z} - er^{4z} - \&c.$$



Ubi r, a, b, c, d, e, &c. ſunt quantitates invariabiles; atque erit diſtantia inter Abſciſſam & Aſymptoton, hoc eſt,

$$PQ = A +$$
$$\frac{A - B}{r - 1} +$$
$$\frac{rA - \overline{r+1}B + C}{\overline{r-1}.\overline{r^2-1}} +$$
$$\frac{r^3A - \overline{r^3+r^2+r}B + \overline{r^2+r+1}C - B}{\overline{r-1}.\overline{r^2-1}.\overline{r^3-1}} +$$
$$\frac{r^6A - \overline{r^6+r^5+r^4+r^3}B + \overline{r^5+r^4+2r^3+r^2+r}C - \overline{r^3+r^2+r+1}D + E}{\overline{r-1}.\overline{r^2-1}.\overline{r^3-1}.\overline{r^4-1}} +$$
&c.

Coefficientes literarum A, B, C, D, &c. in diverſis Terminis, formantur per continuam multiplicationem numerorum 1, $\frac{n-1}{r-1}$, $\frac{n-r}{r^2-1}$, $\frac{n-r^2}{r^3-1}$, $\frac{n-r^3}{r^4-1}$, &c. In primo autem Termino eſt $n = r^0$, in ſecundo $n = r^1$, in tertio $n = r^2$, in quarto $n = r^3$, & ſic porro. Exempli gratiâ, in quarto Termino Coefficientes ſumpti ordine inverſo ſunt 1, $1 + r + r^2$, $r + r^2 + r^3$, r^3, neglectis ſignis; eſt vero $1 \times \frac{r^3 - 1}{r - 1} = 1 + r + r^2$, $\overline{1 + r + r^2} \times \frac{r^3 - r}{r^2 - 1} = r + r^2 + r^3$, denique $\overline{r + r^2 + r^3} \times \frac{r^3 - r^2}{r^3 - 1} = r^3$; ſic itaque inveniuntur Coefficientes. In hâc autem Propoſitione exci-

pio caſum in quo eſt $r = \pm 1$; tunc enim degenerat Hyperbola in lineam rectam.

Series vero ſic inveſtigatur. In Æquatione aſſumptâ $y = a - br^z - cr^{2z} -$ &c. mutetur ſignum Abſciſſæ z, & prodibit $y = a - \frac{b}{r^z} - \frac{c}{r^{2z}} - \frac{d}{r^{3z}} -$ &c. evadat jam z infinite magna, atque in valore Ordinatæ y evaneſcent omnes Termini præter primum dummodo concipiatur r eſſe major unitate; eoque pacto erit $y = a$; hoc eſt, Ordinata ad diſtantiam infinitam remota, ſive diſtantia inter Abſciſſam & Aſymptoton PQ æqualis eſt primo Termino a: nam in diſtantiâ infinitâ coincidit Curva cum ſuâ Aſymptoto. Quantitas autem a ſic inveſtigatur in Æquatione prius aſſumptâ $y = a - br^z - cr^{2z} - dr^{3z} - er^{4z} -$ &c. ſcribe Ordinatas æquidiſtantes A, B, C, D, E, &c. ſucceſſive pro y, & interea 0, 1, 2, 3, 4, &c. pro Abſciſſâ z: ac prodibunt Æquationes

$$A = a - b - c - d - e - \&c.$$
$$B = a - br - cr^2 - dr^3 - er^4 - \&c.$$
$$C = a - br^2 - cr^4 - dr^6 - er^8 - \&c.$$
$$D = a - br^3 - cr^6 - dr^9 - er^{12} - \&c.$$
$$E = a - br^4 - cr^8 - dr^{12} - er^{16} - \&c.$$
&c.

Habentur igitur tot Æquationes quot ſunt incognitæ a, b, c, d, e, &c. ex quibus quæratur a per Algebram vulgarem, atque ejus valor prodibit idem ac jam ipſi PQ aſſignatus eſt. Q. E. D.

Corollarium. Hinc ſi dentur aliquot Termini initiales in Serie infinitâ, quorum differentiæ ſunt proxime in progreſſione Geometricâ, dabitur ultimus omnium utique qui removetur ad diſtantiam infinitam ab initio. Nam ſi A, B, C, D, &c. ordine inverſo ſumpti, deſignent Terminos quorum differentiæ ſunt fere in proportione Geometricâ ut r^0, r^1, r^2, r^3, &c. ultimus omnium æqualis erit PQ intervallo inter Abſciſſam & Aſymptoton: vel ſtylo *Gregoriano* dabitur Terminatio Seriei.

Exemplum.

Datis Polygonis aliquot regularibus Circulo inſcriptis, invenire ultimum Polygonorum ſive Aream Circuli. Ea ſunto

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4	2.00000.00000.0000 = F	3.14033.11569.5475
8	2.82842.71247.4619 = E	126.08888.0294
16	3.06146.74589.2072 = D	6072.7439
32	3.12144.51522.5805 = C	5.5652
64	3.13654.84905.4594 = B	119
128	3.14033.11569.5475 = A	
		3.14159.26535.8979

Dicatur jam ultimum Polygonum A, penultimum B, antepenultimum C, & ſic retro. Et quoniam eorum differentiæ A—B, B—C, C—D, &c. ſunt quamproxime ut Termini 1, 4, 16, 64, 256, &c. id eſt, ut dignitates Quaternarii, erit $r=4$; quo ſubſtituto, Series generalis evadit

$$A + \frac{A-B}{3} + \frac{4A-5B+C}{3.15} + \frac{64A-84B+21C-D}{3.15.63} + \&c.$$

In qua ſcribe pro A, B, C, &c. ſuos valores, & primi quinque Termini dabunt Aream Circuli ad quindecem loca figurarum, ut ex computo appoſito manifeſtum eſt. Et ſimiliter res conſit per Polygona circumſcripta.

Hâc autem ratione quævis Series ſummari poteſt. Nam ſi Ordinatæ æquidiſtantes denotent ſummas ſucceſſivas, valor totius Seriei æquabitur diſtantiæ inter Aſymptoton & Abſciſſam. Si Series ſummanda ſit hujuſmodi $a+bx+cx^2+dx^3+$&c. ubi x, x^2, x^3, &c. denotant partes Terminorum quæ ſunt in progreſſione Geometrica, erit $r=x$; & valor ipſius PQ converget eo celerius quo longius Summæ ſucceſſivæ diſtant a principio, quæ denotantur per A, B, C, D, &c. In caſu autem quando eſt $r=\pm 1$, aſſumenda eſt Hyperbola quæ definitur hujuſmodi Æquatione $y=\frac{1}{z^n}$ in $a+\frac{b}{z}+\frac{c}{z^2}+\frac{d}{z^3}+$&c. vice Hyperbolæ Logarithmicæ; atque index n determinabitur ex naturâ Seriei ſummandæ.

Notandum eſt Series infinitas æque ſummari poſſe per Parabolam *Newtoni* ac per haſce Hyperbolas. Nam Ordinatæ quæ in Hyperbolis ſunt æquidiſtantes, ſi ad certas quaſdam diſtantias conſtituantur; exhibebunt, ope Parabolæ, eaſdem expreſſiones pro valoribus Serierum.

Numerus figurarum quæ veræ ſunt in Polygono A, duplicantur per duos Terminos Seriei, triplicantur per tres, & ita deinceps. Sic in Exemplo allato ſunt tres veræ figuræ 314 in Polygono A; & inde quinque Termini dederunt Aream Circuli ad quindecem figurarum loca. Et hæ ſunt Approximationes quales olim invenerunt *Jacobus Gregorius* & *Hugenius:* hic quidem triplicavit veras figuras, at ille eaſdem quadruplicavit, quintuplicavit, imo ſine limite produxit, uti videre eſt in Appendice ad *Veram Circuli & Hyperbolæ Quadraturam.*

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Propositio XXXI.

Invenire Aream cujuſvis Curvæ quamproxime ex datis aliquot ejus Ordinatis æquidiſtantibus.

Per extremitates Ordinatarum deſcribe figuram Parabolicam, ejuſque Area quæ invenitur per methodos notas, proxime æquabitur Areæ Curvæ propoſitæ. Q. E. I.

Scholion.

Quoniam laborioſum eſſet ſemper recurrere ad Parabolam, computavi Tabulam ſequentem quæ exhibet Aream Curvæ directe ex datis aliquot ejus Ordinatis æquidiſtantibus.

Tabula Arearum.

3	$\frac{A+4B}{6}R$
5	$\frac{7A+32B+12C}{90}R$
7	$\frac{41A+216B+27C+272D}{840}R$
9	$\frac{989A+5888B-928C+10496D-4540E}{28350}R$

Tabula Correctionum.

3	$\frac{P-4A+6B}{180}R$
5	$\frac{P-6A+15B-20C}{470}R$
7	$\frac{P-8A+28B-56C+70D}{930}R$
9	$\frac{P-10A+45B-120C+210D-252E}{1600}R$

In hiſce Tabulis A eſt ſumma primæ & ultimæ Ordinatæ, B ſecundæ & penultimæ, C tertiæ & antepenultimæ, & ſic porro uſque dum deventum

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deventum fuerit ad Ordinatam in medio omnium quæ per ultimam literarum A, B, C, &c. repreſentatur. R eſt baſis ſupra quam jacet Area, ſeu pars Abſciſſæ inter primam & ultimam Ordinatam intercepta. P eſt ſumma duarum Ordinatarum quarum una conſiſtit ante primam, altera poſt ultimam ad diſtantias æquales intervallo communi reliquarum Ordinatarum. Numerus autem Ordinatarum qui hic eſt impar, ſignatur ad latera Tabularum. Expreſſiones in Tabula Arearum ſunt Areæ contentæ inter Baſin, Curvam & Ordinatas hinc inde extremas. Eæ vero in Tabula Correctionum ſunt ejuſdem circiter magnitudinis ac differentiæ inter Areas veras & eas per Tabulam prodeuntes: adeoque ſi prima figura Correctionis inveniatur, dein adjiciatur ubi Correctio eſt negativa, vel ſubducatur quando eadem eſt affirmativa; concludere tuto licet Aream ſic correctam, veram eſſe in eo loco decimalium in quo intrat prima figura Correctionis, nec ultra. Itaque per Tabulam Correctionum Area inventa corrigitur, & ſimul numerus figurarum verarum dignoſcitur.

Exemplum.

Sit $\frac{1}{1+x}$ Ordinatæ Hyperbolæ æquilateræ, & quæratur Area ejus quæ jacet ſupra Abſciſſam unitati æqualem. Pro x ſcribe ſucceſſive $\frac{0}{8}, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, \frac{8}{8}$, & provenient novem Ordinatæ

$$\frac{8}{8}, \frac{8}{9}, \frac{8}{10}, \frac{8}{11}, \frac{8}{12}, \frac{8}{13}, \frac{8}{14}, \frac{8}{15}, \frac{8}{16}$$

Igitur eſt $A=\frac{8}{8}+\frac{8}{16}=\frac{3}{2}$, $B=\frac{8}{9}+\frac{8}{15}=\frac{64}{45}$, $C=\frac{8}{10}+\frac{8}{14}=\frac{48}{35}$, $D=\frac{8}{11}+\frac{8}{13}=\frac{192}{143}$, $E=\frac{8}{12}=\frac{2}{3}$: & hiſce ſubſtitutis in ultima expreſſione Arearum, & unitate pro R, oritur Area .69314721. Deinde in Ordinata $\frac{1}{1+x}$ ſcribe $\frac{-1}{8}$ & $+\frac{9}{9}$ ſucceſſive pro x, & provenient duæ Ordinatæ $\frac{8}{7}$ & $\frac{8}{17}$, quarum prior ſtat ante primam, & poſterior poſt ultimam; adeoque $P=\frac{8}{7}+\frac{8}{17}=\frac{192}{119}$: quem ſubſtitue pro P; & pro A, B, C, D, E, ſuos valores, & Correctio pro novem Ordinatis dabit $+$ .00000003, quæ quum affirmativa ſit, ſubducatur a valore Areæ prius invento, & reſtabit .69314718 accurata in ultima figura.

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Computaveram haſce Tabulas ulterius, ceterum Expreſſiones pro undecem aut pluribus Ordinatis ſunt uſibus inepti propter immenſam magnitudinem Coefficientium numeralium. Sin vero novem Ordinatæ non dent Aream ſatis accuratam; dividatur baſis in duas vel plures partes; & inde Area dividetur in totidem, quarum quamque ſi quæras ſeorſim per novem Ordinatas, habebis totam Aream pro lubitu accuratam. Sed & nonnunquam convenit quærere partem Areæ per Seriem infinitam, præſertim ſi Curva decuſſet baſin in angulo recto. Atque hiſce præcognitis, Area quævis habebitur ſatis accurate per Tabulam jam appoſitam.

Sed & Areæ Curvarum haud incommode exprimi poſſint per Differentias Ordinatarum æquidiſtantium ad modum ſequentem.

Tabula Arearum per Differentias Ordinatarum.

1	A
3	$A+\frac{1}{6}B$
5	$A+\frac{2}{3}B+\frac{7}{90}C$
7	$A+\frac{3}{2}B+\frac{11}{20}C+\frac{41}{840}D$
9	$A+\frac{8}{3}B+\frac{86}{45}C+\frac{92}{189}D+\frac{3}{86}E$
11	$A+\frac{25}{6}B+\frac{175}{36}C+\frac{3445}{1512}D+\frac{4045}{9072}E+\frac{94}{3503}F$
13	$A+6B+\frac{103}{10}C+\frac{158}{81}D+\frac{1833}{700}E+\frac{4813}{11550}F+\frac{66}{3050}G$

In hac Tabula A eſt Ordinata in medio omnium, B eſt differentia ſecunda trium Ordinatarum in medio, C eſt differentia quarta quinque Ordinatarum in medio, & ſic porro ad ultimam literarum A, B, C, D, E, F, G, quæ eſt ultima differentia omnium Ordinatarum. Ut ſi ſint quinque Ordinatæ a, b, c, d, e; erit $A=c$, $B=b-2c+d$, $C=a-4b+6c-4d+e$. Et ſic in aliis caſibus. Expreſſiones autem ductæ in baſin Curvæ, ſive partem Abſciſſæ contentam inter primam & ultimam Ordinatam, dant Areas ex dato numero Ordinatarum qui ſignatur ad latus. Notandum eſt ultimos Terminos in expreſſionibus pro novem, undecem & tredecem Ordinatis, non eſſe veros ſed hiſce ſimpliciores & vero ſatis proximos. Nam Ordinata media A & differentiæ B, C, D, E, &c. conſtituunt Seriem convergentem; ideoque non

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non requiritur ut Coefficientes ultimorum Terminorum qui intrant computum, ſint præciſe accurati. Ex convergentia autem Seriei A, B, C, D, &c. dignoſcitur ad quot figuras Area exhibetur accurata; ideoque hæc Tabula non indiget Tabulâ Correctionum. Sed & Coefficientes numerales ſunt longe magis exigui quam in Tabula ſuperiore, atque eâ de cauſâ hæc præferenda eſt, præſertim in magno Ordinatarum numero.

Quæratur rurſus Area Hyperbolæ, cujus Ordinata eſt $\frac{1}{1+x}$; pro x ſcribe o, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$; & provenient quinque Ordinatæ æquidiſtantes 1, $\frac{4}{5}$, $\frac{2}{3}$, $\frac{4}{7}$, $\frac{1}{2}$: erit igitur $A=\frac{2}{3}$, nempe Ordinatæ mediæ: $B=\frac{4}{105}$, ſcilicet differentiæ ſecundæ trium Ordinatarum in medio $\frac{4}{5}$, $\frac{2}{3}$, $\frac{4}{7}$: denique $C=\frac{1}{70}$, hoc eſt, æqualis ultimæ differentiæ omnium. Et hi valores ſubſtituti in Expreſſione pro quinque Ordinatis dant $\frac{2}{3}+\frac{8}{315}+\frac{1}{900}=\frac{4367}{6300}$, quæ ducta in baſin ſive unitatem & reducta ad decimales, evadit ,69317 pro Areâ quæſita. Et hæc Area ad minimum juſta eſt in quarto decimalium loco, in quo ſcilicet intrat prima figura ultimi Termini $\frac{1}{900}$.

Sed ut differentiæ promptius ac facilius inveniantur; ſit a Ordinata media, b ſumma duarum hinc inde mediæ proximarum, dein c ſumma duarum quæ hinc inde ſequuntur, & ſic porro; tum pone

$$A=a,$$
$$B=b-2A,$$
$$C=c-2A-4B,$$
$$D=d-2A-9B-6C,$$
$$E=e-2A-16B-20C-8D,$$
$$F=f-2A-25B-50C-35D-10E,$$
$$G=g-2A-36B-105C-112D-54E-12F,$$

Atque A, B, C, D, &c. erunt Ordinata media, differentia ſecundâ trium mediarum, quarta quinque mediarum, & ſic porro in reliquis reſpective.

Propositio XXXII.

Deſignent a, b, c, d, e, f, &c. Terminos æquidiſtantes continue tendentes ad rationem æqualitatis, & Æquationes ſequentes approximabunt ad eorumdem relationes.

2	$a-b=0$
3	$a-2b+c=0$
4	$a-3b+3c-d=0$
5	$a-4b+6c-4d+e=0$
6	$a-5b+10c-10d+5e-f=0$
7	$a-6b+15c-20d+15e-6f+g=0$
8	$a-7b+21c-35d+35e-21f+7g-h=0$
9	$a-8b+28c-56d+70e-56f+28g-8h+i=0$
10	$a-9b+36c-84d+126e-126f+84g-36h+9i-k=0$

&c.

Hæc Tabula in uſum reſervanda eſt, ut conſulatur quoties opus ſit. Patet autem Coefficientes numerales eſſe Uncias dignitatum binomii. Et demonſtratio patet: quoniam enim Termini ſupponuntur continue tendere ad rationem æqualitatis, erunt eorum differentiæ $a-b$, $b-c$, $c-d$, $d-e$, &c. parvæ; deinde differentiæ differentiarum $a-2b+c$, $b-2c+d$, $c-2d+e$, &c. erunt minores differentiis primis; atque tertiæ $a-3b+3c-d$, $b-3c+3d-e$, &c. erunt minores ſecundis; & quartæ $a-4b+6c-4d+e$, &c. erunt minores tertiis, & ſic in infinitum. Ergo differentiæ primæ, ſecundæ, tertiæ, reliquæque poſitæ æquales nihil ut in Propoſitione, continue approximabunt ad veram relationem Terminorum. Q. E. D.

Corollarium. Hinc in Serie æquidiſtantium, ſi deſit Terminus quivis, poteſt is inveniri per hanc Propoſitionem. Ut ſi ſint quinque Termini a, b, c, d, e; eorum relatio erit $a-4b+6c-4d+e=0$; & ex hâc Æquatione dabitur eorum quilibet quamproxime ex datis reliquis.

Et notandum eſt Terminum quemvis, ceteris paribus, eo accuratius definiri, quo propius conſtet medio omnium: atque errores a vero eſſe quamproxime ut Coefficientes numerales Terminorum quæſitorum reciproce. Conſiſtat igitur Terminus quæſitus vel in medio omnium vel eidem quantum fieri poteſt proximus.

Exemplum.

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Exemplum.

Quæratur Logarithmus numeri 53 ex datis Logarithmis numerorum aliquot antecedentium. Pone a pro Logarithmo quæſito, eritque

$$l,52=b=1.71600.33436$$
$$l,51=c=1.70757.01761$$
$$l,50=d=1.69897.00043$$
$$l,49=e=1.69019.06800$$
$$l,48=f=1.68124.12374$$
$$l,47=g=1.67209.78579$$

Deinde relatio inter ſeptem Terminos $a, b, c, d, e, f, g,$ erit $a-6b+15c-20d+15e-6f+g=0$, adeoque $a=6b-15c+20d-15e+6f-g$: ubi ſubſtituendo pro $b, c, d, e, f, g,$ ſuos valores, habebitur 1.72427.58726 pro a ſive Logarithmo numeri 53, exiſtente .00000.00030 errore in exceſſu. Ceterum ſi dentur ſex Logarithmi quorum tres conſiſtunt ante, & tres reliqui poſt illum qui quæritur; in eo inquam caſu Logarithmus quæſitus accuratiſſime definietur.

Igitur deſignent A, B, C, D, &c. ſummas datorum Terminorum qui hinc inde æqualiter diſtant a quæſito, & ejus valores erunt ut in Tabula ſequente

2	$\frac{A}{2}$
4	$\frac{4A-B}{6}$
6	$\frac{15A-6B+C}{20}$
8	$\frac{56A-28B+8C-D}{70}$
10	$\frac{210A-120B+45C-10D+E}{252}$
	&c.

Dentur verbi gratiâ Logarithmi numerorum 50, 51, 52, 54, 55, 56, & oporteat invenire Logarithmum numeri 53. Pone

$$l,52+l,54=3.44839.71034$$
$$l,51+l,55=3.44793.28656$$
$$l,50+l,56=3.44715.80313$$

Dein

Dein pro A, B & C ſubſtitue hoſce valores in Expreſſione pro ſex Terminis, & prodibit 1.72427.58695 pro Logarithmo numeri 53, exiſtente unitate errore in ultima figura. Hinc in Tabulis Logarithmicis, Trigonometricis, Aſtronomicis, aliiſque ejuſmodi; ſi deſit forte Terminus quivis, poteſt is inſeri per hanc Propoſitionem; vel ſi ſuſpicio ſit Terminum quemvis erroneum eſſe, poteſt eâdem methodo corrigi. Nam Expreſſiones hic exhibitæ ſunt generales, utique quæ non dependent ex natura Tabulæ cujuſvis particularis.

Propositio XXXIII.

Deſignent &c. $\varepsilon, \delta, \gamma, \beta, \alpha, a, b, c, d, e$, &c. Terminos alternos in Serie utrinque excurrente in infinitum, & ponatur $A = a + \alpha$, $B = b + \beta$, $C = c + \gamma$, $D = d + \delta$, $E = e + \varepsilon$, & ſic porro; atque Terminus in medio inter a & α conſiſtens, æqualis erit

$$\frac{1}{2} \times A +$$
$$\frac{1}{16} \times \overline{A - B} +$$
$$\frac{3}{256} \times \overline{2A - 3B + C} +$$
$$\frac{5}{2048} \times \overline{5A - 9B + 5C - D} +$$
$$\frac{35}{65536} \times \overline{14A - 28B + 20C - 7D + E} +$$
&c.

Coefficientes numerales literarum A, B, C, D, &c. ſunt differentiæ Unciarum in diverſis dignitatibus Binomii: Et Coefficientes qui ducuntur in totos Terminos, ſcilicet $\frac{1}{2}$, $\frac{1}{16}$, $\frac{3}{256}$, &c. generantur per continuam multiplicationem numerorum $\frac{1}{4.2}$, $\frac{3}{4.4}$, $\frac{5}{4.6}$, $\frac{7}{4.8}$, &c. Hiſce vero præcognitis, Series ad libitum producitur.

Et Series inveſtigatur ponendo $z = 0$ in Caſu ſecundo Propoſitionis vigeſimæ; tunc enim habebitur Ordinata ſive Terminus qui conſiſtit in medio omnium.

Poſſint

Poſſint autem Termini Seriei in unam ſummam colligi ut hic vides,

2	$\frac{A}{2}$
4	$\frac{9A-B}{16}$
6	$\frac{150A-25B+3C}{256}$
8	$\frac{1225A-245B+49C-5D}{2048}$
10	$\frac{39690A-8820B+2268C-405D+35E}{65536}$

Prima Expreſſio eſt primus Terminus Seriei, ſecunda eſt Summa primi & ſecundi, tertia eſt Summa trium primorum Terminorum, & ſic deinceps. Ex datis itaque Terminis alternis, intermedii confeſtim dabuntur per hanc Tabulam vel per ipſam Seriem. Prima Expreſſio $\frac{A}{2}$ ſufficit quando ſecundus Terminus Seriei minor eſt quam qui ingrediatur computum. Et ſimiliter in reliquis; nam Termini Seriei ſunt differentiæ inter Expreſſiones & veritatem quamproxime: adeoque ſemper ſcire licet quænam expreſſio ſufficit propoſito.

Exempli gratiâ, ſi quæratur Logarithmus numeri 53 ex datis iis numerorum 46, 48, 50, 52, 54, 56, 58, 60; pone

$$l, 52+l, 54=A=3.44839.71035$$
$$l, 50+l, 56=B=3.44715.80313$$
$$l, 48+l, 58=C=3.44466.92309$$
$$l, 46+l, 60=D=3.44090.90820.$$

Et hiſce valoribus ſcriptis in Serie; vel in Expreſſione pro octo Terminis, habebitur 1.72427.58696 pro Logarithmo numeri 53. Et eâdem ratione invenire licet quemvis alium intermedium. In Conſtructione ergo Tabularum ſufficit primo quærere aliquos Terminos in debitis diſtantiis; nam reliqui poſſunt hâc methodo inſeri. Etenim continue intercalandi ſunt Termini primo inventi, uſque dum perventum fuerit ad ultimos qui ingrediuntur Tabulam. Et notandum eſt, oportere omnes Terminos computari circa initium Tabulæ propter magnas differentias; dein per ſaltum pertranſire licet alternos decreſcentibus differentiis; & poſtea ternos, ſeptenoſque ubi differentiæ ſunt minores. Et hæc eſt methodus quam præcipit *Newtonus:* ceterum præferendæ ſunt regulæ particulares deductæ ex natura Tabulæ conſtruendæ; etenim hæ minore labore plerumque abſolvent opus.

F I N I S.

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ERRATA.

PAG. 4. l. 5. pro $z+\frac{2}{3}$, lege $z+\frac{3}{2}$.

Pag. 8. l. 18. pro $z+4$, lege $z-4$.

Pag. 44. l. 8. pro $T'\times\frac{z-m}{z}$, lege $T\times\frac{z-m}{z}$.

Pag. 67. l. 12. pro $B=\frac{2.3.3}{14.29.29}B$, lege $C=\frac{2.3.3}{14.29.29}B$.

Pag. 76. l. 26. pro $y+xy$, lege $y+x\dot{y}$.

Pag. 80. l. 13. pro $2xyy$, lege $2xy\dot{y}$.

Pag. 83. l. pen. pro eritque, lege erit.

Pag. 91. l. 21. dele &c.

Pag. 94. l. 12. pro $\frac{1}{1999}$, lege $\frac{1}{2000}$.

Pag. 115. l. 12. pro media, lege medio.

Pag. 120. l. 3. pro periferiæ, lege ſemiperiferiæ.

Pag. 122. l. 15. pro 46, lege 4, 6.

Pag. 136. l. 28. pro $x+\frac{1}{2}=10$, lege $x+\frac{1}{2}=11$.

Pag. 137. l. 5. pro $r=\frac{1}{2}$, lege $n=\frac{1}{2}$.

Ibid. l. 21. pro $x=501$, lege $x=501\frac{1}{2}$.

Pag. 147. l. 18. pro Ordinatæ, lege Ordinata.

Ibid. l. 25. pro $\frac{9}{9}$, lege $\frac{9}{8}$.

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